MIXED BOUNDARY VALUE PROBLEMS OF POTENTIAL THEORY AND THEIR APPLICATIONS IN ENGINEERING

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INTRODUCTION

This book may be considered as logical continuation of the previously published (V.I. Fabrikant, *Applications of Potential Theory in Mechanics*, Kluwer Academic, 1989), where a new and elementary method was described for solving mixed boundary value problems. The method can solve *non-axisymmetric problems* as easily as axisymmetric ones, *exactly and in closed form*. It enables us to treat *analytically* non-classical domains. The method also provides, as a bonus, a tool for exact evaluation of various two-dimensional integrals involving distances between two or more points.

The main emphasis of the first book was on solid mechanics problems. Here, we describe various applications of the new method to electromagnetics, acoustics and diffusion. Also included in this book are some results in fracture mechanics and elastic contact problems which were obtained just recently and could not be included in the first book.

The book is addressed to a wide audience ranging from engineers to mathematical physicists. While an engineer can find in the book some elementary, ready to use formulae for solving various practical problems, a mathematical physicist might become interested in new applications of the mathematical apparatus presented. The book should be of interest to specialists in electromagnetics, acoustics, diffusion, solid and fluid mechanics, etc.

The book is *accessible to anyone* with a background in university undergraduate calculus, but should be of interest to established scientists as well. Though the method is elementary, the transformations involved are sometimes very non-trivial and cumbersome, while the final result is usually very simple. The reader who is interested only in application of the general results to his/her particular problems may skip the long derivations and use the final formulae which requires little effort. The reader who wants to master the method in order to solve new problems has to repeat the derivations which are given in sufficient detail. The exercises are important in this regard.

The book is based entirely on the author's results, and this is why the work of other scientists is mentioned only when such a quotation is inevitable for some reason, like numerical data needed to verify the accuracy of approximate results, comparison with existing results, or pointing out some errors in publications. There are several books and review articles presenting an adequate account of the state-of-the-art in the field. Appropriate references are given for the reader's convenience. The purpose of this book was neither to repeat nor to compete with them.

For the reader's convenience, it was attempted to make each chapter (and section, wherever possible) self-contained. The reader can skip several sections and continue reading, without loosing the ability to understand material. On the other hand, this resulted in repetition of some definitions and descriptions. The author thinks that the additional convenience is worth several extra pages in the book.

A new and more general definition of the $\mathcal{L}$-operator is given in Chapter 1. This definition gives rigorous justification to the mathematical formalism involved. Various forms of integral representation for the reciprocal of the distance between two points follows. A general solution is presented to basic mixed boundary value problems for a half-space in cylindrical coordinates. These results are generalized for the case of spherical and toroidal coordinates.

We can generalize the Newton potential as $V = H/R^{1+\kappa}$, where R is the distance between two points, H is a constant depending on the physical properties of the space, and $-1 < \kappa < 1$. This potential has various applications in engineering, for example, in the theory of elasticity of inhomogeneous elastic body, with the modulus of elasticity E being a power function of z: $E = E_0 z^{\kappa}$. Other applications include fluid mechanics and heat transfer. Closed form solution to various non-axisymmetric problems is given in Chapter 2.

The general results of Chapter 1 are applied in Chapter 3 to investigation of interaction of several charged coaxial and arbitrarily located disks. New type of governing integral equation is derived for the Dirichlet and Neumann problems for a circular annulus domain. Simple yet accurate formulae are derived for the capacity of flat laminae. Similar results were obtained for the electrical and magnetic polarizability of small apertures of general shape.

Advances in bioengineering have generated wide interest in the diffusion mechanism of biological membranes. The diffusion process through a thin membrane, perforated by several holes of arbitrary shape, is considered in Chapter 4. A general theorem is established which relates the total flux through each hole, with the concentration distribution of some chemical species prescribed in the hole, to a system of linear algebraic equations. The theorem is applied to the case of arbitrarily located circular and elliptical holes. The influence of the pore length is investigated by a new method. Application of the main results to the problems of sound transmission through an arbitrary aperture in a soft and rigid screen is also presented.

Chapter 5 contains complete solutions to several contact problems which were obtained recently, and could not be included in (Fabrikant, 1989). Those comprise complete elastic fields around axisymmetric and inclined bonded punch. These fundamental solutions allow us to solve various problems of interaction between punches and anchor loads. Two of such solutions are included. A new approach is presented to a general annular punch problem, with analytical, numerical and asymptotic solutions derived and compared.

Some new results in fracture mechanics are presented in Chapter 6. A complete solution is given for the first time to the case of general antisymmetric loading of internal and external circular cracks. All the relevant Green's functions are given explicitly and in closed form. A superposition of antisymmetric solution with a symmetric allows us to solve the problem of one-sided loading of a crack as well as various interactions between a crack and a general external force. A new approach is presented to the problem of a flat crack in the shape of a circular annulus, subjected to a general normal load. The problem is reduced to two two-dimensional integral equations with elementary kernels. The equations are non-singular, and can be easily uncoupled. An accurate numerical solution can be obtained by any standard method.

Chapter 7 is devoted to the numerical methods of solution of one-dimensional integral and integro-differential equations. The solution is represented in the form of power series with undetermined coefficients multiplied by a function in which the essential features of the singularity of the solution are preserved. The method of collocations is used to determine the unknown coefficients. The examples show that the method suggested is more general and gives good results even in the case when the form of solution does not exactly preserve the essential features of singularity. The method is simpler than others which use the properties of orthogonal polynomials. A standard FORTRAN subroutine is presented for solving general one-dimensional integro-differential equations. An algorithm and a standard subroutine are developed for computer evaluation of two-dimensional singular integrals. The software is used in numerical analysis of various non-classical two- and three-dimensional contact problems.

The book contains so much new material that some misprints and errors are inevitable, though every effort was made to eliminate them. The author would be grateful for every communication in this regard. All the readers' comments are welcome.

CHAPTER 1

NEW RESULTS IN POTENTIAL THEORY

1.1. The $\mathcal{L}$-operator and its properties

Let $f(r,\phi)$ be an arbitrary function which belongs to $L^1[0,2\pi]$ as a function of ϕ for any fixed $r\geq 0$. Let us associate with $f(r,\phi)$ the sequence

$$\{f_n(r)\} \qquad -\infty < n < \infty \tag{1.1.1}$$

of its Fourier coefficients. Consider $\mathcal{L}(k)$ as an operator on the linear space of sequences $\{f_n(r)\}$. We do not define any topology on this space. The algebraic operations are defined naturally as follows:

$$c_1\{f_n^{(1)}\} + c_2\{f_n^{(2)}\} := \{c_1 f_n^{(1)} + c_2 f_n^{(2)}\}, \tag{1.1.2}$$

$$\{f_n^{(1)}\} = \{f_n^{(2)}\} \iff f_n^{(1)} = f_n^{(2)} \quad \forall\, n \tag{1.1.3}$$

We define

$$\mathcal{L}(k)\{f_n\} = \{k^{|n|} f_n\}. \tag{1.1.4}$$

This definition makes sense for any $k\in\mathbb{C}$, and it implies that

$$\mathcal{L}(k_1)\,\mathcal{L}(k_2)\{f\} = \mathcal{L}(k_1 k_2)\,\{f\} \quad \forall\ k_1, k_2 \in \mathbb{C}, \tag{1.1.5}$$

$$\mathcal{L}(1)\,\{f\} = \{f\}. \tag{1.1.6}$$

Equation (6) is a particular case of (5) corresponding to $k_1 = k \neq 0$ and $k_2 = k^{-1}$.

Consider now the operator $\prod_{j=1}^{m} \mathcal{L}(k_j)$. It is well defined for any k_j, in particular, in the case when some of the k_j are greater than 1. An obvious corollary is: if $|\prod_{j=1}^{m} k_j| < 1$ then $\prod_{j=1}^{m} \mathcal{L}(k_j)\{f_n\}$ is a sequence of the Fourier coefficients of some function belonging to $L^1[0,2\pi]$ if $\{f_n\}$ is a sequence of the Fourier coefficients of a function $f(\phi) \in L^1[0,2\pi]$, and, moreover, the Fourier series corresponding to the sequence $\prod_{j=1}^{m} \mathcal{L}(k_j)\{f_n\}$ converges absolutely and uniformly in $\phi \in [0,2\pi]$.

In the case when $k<1$, formula (4) can be rewritten as

$$\mathcal{L}(k)f(\phi) = \sum_{n=-\infty}^{\infty} k^{|n|} f_n e^{in\phi} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|} e^{in\phi} \int_0^{2\pi} e^{-in\phi_0} f(\phi_0)\, d\phi_0. \tag{1.1.7}$$

Summation in (7) yields

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi-\phi_0)\, f(\phi_0)\, d\phi_0, \tag{1.1.8}$$

where the notation was introduced

$$\lambda(k,\psi) = \frac{1-k^2}{1-2k\cos\psi + k^2}\,. \tag{1.1.9}$$

Note that the $\mathcal{L}$-operator, as it is presented in (8), coincides with the one which was introduced by Poisson for solving the two-dimensional boundary value problem of potential theory for a circle. We are going to use it for solving the relevant three-dimensional problems. Whenever the operator $\mathcal{L}(k)$ is applied, with no limitation $k<1$ assumed, its general definition (4) is valid, allowing the properties (5) and (6) to be used. As soon as it becomes clear that $k \le 1$, the representation (8) becomes valid thus making it possible to write the final result in a closed and simplified form.

1.2. Integral representation for the reciprocal of the distance between two points

It was proven in (Fabrikant, 1971a) that

$$\frac{1}{R^{1+u}} = \frac{1}{(\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0))^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, x^u \mathrm{d}x}{\left[(\rho^2-x^2)(\rho_0^2-x^2)\right]^{(1+u)/2}}. \tag{1.2.1}$$

Here λ is defined by (1.1.9). The identity (1) can be verified by the introduction of a new variable

$$\eta(x) = [(\rho^2-x^2)(\rho_0^2-x^2)]^{1/2}/x, \tag{1.2.2}$$

Substitution of the identities

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = -\frac{\rho^2\rho_0^2-x^4}{x^3\eta}, \qquad \lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0) = -\frac{x\eta}{R^2+\eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x}.$$

in (1) transforms it into

$$\frac{1}{R^{1+u}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^\infty \frac{\eta^{-u}\mathrm{d}\eta}{R^2+\eta^2}. \tag{1.2.3}$$

The integral in (3) can be evaluated by using formula (3.241.4) from (Gradshtein and Ryzhik, 1963), thus proving the identity. All the results above are related to the distance between two points in the plane z=0. We need to generalize them to represent

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{(1+u)/2}}. \tag{1.2.4}$$

One can observe that representation (1) remains valid if we formally substitute ρ and ρ_0 by arbitrary quantities l_1 and l_2. We need to choose them so that

$$\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2 = l_1^2+l_2^2-2l_1l_2\cos(\phi-\phi_0). \tag{1.2.5}$$

This leads to two equations

$$l_1^2 + l_2^2 = \rho^2 + \rho_0^2 + z^2, \qquad l_1 l_2 = \rho\rho_0. \tag{1.2.6}$$

Their solution will take the form

$$l_1(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho+\rho_0)^2 + z^2]^{1/2} - [(\rho-\rho_0)^2 + z^2]^{1/2}\}, \tag{1.2.7}$$

$$l_2(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho+\rho_0)^2 + z^2]^{1/2} + [(\rho-\rho_0)^2 + z^2]^{1/2}\}. \tag{1.2.8}$$

Hereafter the following abbreviations will be used:

$$l_1(x) \equiv l_1(x,\rho,z), \quad l_2(x) \equiv l_2(x,\rho,z), \tag{1.2.9}$$

$$l_1 \equiv l_1(a,\rho,z), \qquad l_2 \equiv l_2(a,\rho,z). \tag{1.2.10}$$

Note the limiting properties

$$\lim_{z\to 0} l_1(x) = \min(x,\rho), \qquad \lim_{z\to 0} l_2(x) = \max(x,\rho). \tag{1.2.11}$$

In view of the properties above, the representation (1) can be generalized as follows:

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, x^u \mathrm{d}x}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{(1+u)/2}}. \tag{1.2.12}$$

Formula (12) simplifies when u=0

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, \mathrm{d}x}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}}. \tag{1.2.13}$$

Again, one can notice that the integral in (13) may be evaluated as indefinite

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, dx}{\{[l_1^2(\rho_0)-x^2][l_2^2(\rho_0)-x^2]\}^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{\{[l_1^2(\rho_0)-x^2][l_2^2(\rho_0)-x^2]\}^{1/2}}{xR_0}. \tag{1.2.14}$$

The last representation is very important and will be widely used throughout the book.

Another series of useful formulae can be obtained from those above by a simple change of variables, namely,

$$\int \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, dx}{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{1/2}} = \frac{1}{R_0}\tan^{-1}\frac{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{1/2}}{xR_0}, \tag{1.2.15}$$

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, x^u dx}{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{(1+u)/2}}, \tag{1.2.16}$$

$$\frac{1}{R_0} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}} = \frac{2}{\pi}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, dx}{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{1/2}}, \tag{1.2.17}$$

$$\int \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, dx}{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}} = \frac{1}{R}\tan^{-1}\left[\frac{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}}{xR}\right]. \tag{1.2.18}$$

The representations above are useful for solving external mixed boundary value problems.

Several modifications of (14) are available. For example, we can write

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, dx}{(\rho^2-x^2)^{1/2}[\rho_0^2-g^2(x)]^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{(\rho^2-x^2)^{1/2}[\rho_0^2-g^2(x)]^{1/2}}{xR_0}. \tag{1.2.19}$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}. \tag{1.2.20}$$

It is important to notice that the function $g(x)$ is inverse to l_1 for $x^2 < \rho^2$, and is inverse to l_2 for $x^2 > \rho^2 + z^2$. Introduction of a new variable $x = l_1(y)$, $y = g(x)$ transforms (19) into

$$\int \frac{[l_2^2(y) - y^2]^{1/2}}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - l_1^2(y)]} \lambda\left(\frac{l_1^2(y)}{\rho\rho_0}, \phi - \phi_0\right) \mathrm{d}y$$

$$= -\frac{1}{R_0} \tan^{-1} \frac{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - y^2]^{1/2}}{yR_0} \tag{1.2.21}$$

A particular case of (13), when $z = 0$, reads

$$\frac{1}{R} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]^{1/2}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0)\, \mathrm{d}x}{\sqrt{\rho^2 - x^2}\sqrt{\rho_0^2 - x^2}}. \tag{1.2.22}$$

The same result takes another form due to (17)

$$\frac{1}{R} = \frac{2}{\pi}\int_{max(\rho_0,\rho)}^{\infty} \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi - \phi_0)\, \mathrm{d}x}{\sqrt{x^2 - \rho^2}\sqrt{x^2 - \rho_0^2}}. \tag{1.2.23}$$

1.3. Internal mixed boundary value problem

The problem is called internal when the non-zero boundary conditions are prescribed inside a circle.

Problem 1. Let us consider a typical problem solved by our method. We need to find a function V such that

$$\Delta V = 0 \quad \text{in} \quad \mathbb{R}_+^3 := \{x: x_3 > 0\}, \quad x = (x_1, x_2, x_3) \tag{1.3.1}$$

subject to the following boundary conditions at $z = 0$

$$V = v(\rho, \phi), \quad \text{if} \quad \rho < a, \quad \partial V/\partial z = 0 \quad \text{if} \quad \rho > a, \quad \rho := x_1, \ \phi := x_2, \ z := x_3. \tag{1.3.2}$$

$$V(\infty) = 0. \tag{1.3.3}$$

Here (ρ,ϕ) are polar coordinates in the plane $P = \{x\colon\ x_3 = 0\}$; and v is a given function.

The problem can be interpreted as an electrostatic one of a charged disc, with a certain potential prescribed on its surface, or it can be interpreted as an elastic contact problem of a circular punch pressed against an elastic half-space; other interpretations are also possible. We call the problem internal because the non-zero conditions are prescribed inside the disc. The potential function V can be represented through a simple layer as follows:

$$V(\rho,\phi,z) = \int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)}{R_0}\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0 \tag{1.3.4}$$

Here

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}, \quad \text{and} \quad \sigma = -\frac{1}{2\pi}\frac{\partial V}{\partial z}\Big|_{z=0}. \tag{1.3.5}$$

Substitution of (13) in (4) yields, after changing the order of integration

$$V(\rho,\phi,z) = 4\int_0^{l_1} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_{g(x)}^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - g^2(x)}}\, \mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) \tag{1.3.6}$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}, \tag{1.3.7}$$

the $\mathcal{L}$-operator is defined by (1.1.4), the abbreviations l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively; and the following rule is used for changing the order of integration:

$$\int_0^a \mathrm{d}\rho_0 \int_0^{l_1(\rho_0)} \mathrm{d}x = \int_0^{l_1} \mathrm{d}x \int_{g(x)}^a \mathrm{d}\rho_0. \tag{1.3.8}$$

Substitution of the boundary condition (2) in (6) leads to the governing integral equation

$$4\int_0^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}}\, \mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) = v(\rho,\phi). \tag{1.3.9}$$

Expression (9) is now presented as a sequence of two Abel-type operators and one $\mathcal{L}$-operator. We recall that the general Abel integral equation

$$\int_x^a \frac{F(y)\,\mathrm{d}y}{(y^2-x^2)^{(1+u)/2}} = f(x) \tag{1.3.10}$$

has the solution

$$F(r) = -\frac{2\cos(\pi u/2)}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{f(x)\,x\mathrm{d}x}{(x^2-r^2)^{(1-u)/2}}. \tag{1.3.11}$$

Since the variables in the Abel operators of (9) are interwoven with those of the $\mathcal{L}$-operator, we need to apply their combination, in order to invert (9). In view of the new definition of the $\mathcal{L}$-operator (1.1.4), equation (9) can be rewritten as

$$4\int_0^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\left\{\left(\frac{x^2}{\rho\rho_0}\right)^{|n|}\sigma_n(\rho_0)\right\} = \{v_n(\rho)\}. \tag{1.3.12}$$

We have here a sequence of one-dimensional integral equations. The first operator to be applied to both sides of (12) is

$$\mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\,\mathcal{L}(\rho), \tag{1.3.13}$$

with the result

$$2\pi\int_t^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-t^2)^{1/2}}\,\mathcal{L}\left(\frac{t}{\rho_0}\right)\{\sigma_n(\rho_0)\} = \mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\,\mathcal{L}(\rho)\,\{v_n(\rho)\}. \tag{1.3.14}$$

The second operator to be applied to both sides of (14) is

$$\mathcal{L}(y)\,\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t\mathrm{d}t}{(t^2-y^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{t}\right),$$

with the result

$$\{\sigma_n(y)\} = -\frac{1}{\pi^2 y}\mathcal{L}(y)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t\mathrm{d}t}{(t^2-y^2)^{1/2}}\mathcal{L}\left(\frac{1}{t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\mathcal{L}(\rho)\{v_n(\rho)\}. \tag{1.3.15}$$

Taking into consideration that $(\rho y/t^2)<1$, the rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (15) as follows:

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{1/2}} - \int_y^a \frac{\mathrm{d}t}{(t^2-y^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{1.3.16}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\mathcal{L}\left(\frac{\rho y}{t^2}\right)v(\rho,\phi)\right]. \tag{1.3.17}$$

Using integration by parts and the fact that $\lambda(k,\psi)$ satisfies the two-dimensional Laplace equation in polar coordinates, the following identity can be established

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi) = \int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\mathcal{L}\left(\frac{\rho y}{t^2}\right)\Delta v(\rho,\phi), \tag{1.3.18}$$

where Δ is the two-dimensional Laplace operator in polar coordinates. Substitution of (18) in (16) leads to another form of solution, namely,

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{1/2}} - \int_y^a \frac{\mathrm{d}t}{(t^2-y^2)^{1/2}}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\mathcal{L}\left(\frac{\rho y}{t^2}\right)\Delta v(\rho,\phi)\right], \tag{1.3.19}$$

Interchange of the order of integration in (19) and integration with respect to t yields

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left\{\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{1/2}}\right.$$

$$\left.-\frac{1}{2\pi}\int_0^{2\pi}\int_0^a \tan^{-1}\left[\frac{\sqrt{a^2-\rho^2}(a^2-y^2)^{1/2}}{a[\rho^2+y^2-2\rho y\cos(\phi-\psi)]^{1/2}}\right]\frac{\Delta v(\rho,\psi)\,\rho\mathrm{d}\rho\mathrm{d}\psi}{[\rho^2+y^2-2\rho y\cos(\phi-\psi)]^{1/2}}\right\}.$$

(1.3.20)

The solution obtained here consists of two parts: the first part is singular at the boundary while the second one vanishes at the boundary. In various applications it is required that the solution be nonsingular at the boundary. The necessary and sufficient condition then is $\Phi(a,a,\phi)=0$. In elastic contact problems this condition defines the radius of the contact domain. Notice also that in the case when v is a two-dimensional harmonic function, the non-trivial solution is singular.

Now it is of interest to express the potential V in the half-space directly through its value v prescribed inside the disc $\rho=a$. Substitution of (15) in (6) yields, after subsequent integration

$$V(\rho,\phi,z)=\frac{2}{\pi}\int_0^{l_1}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\,\mathcal{L}\!\left(\frac{x^2}{\rho g^2(x)}\right)\frac{\mathrm{d}}{\mathrm{d}g(x)}\int_0^{g(x)}\frac{r\,\mathrm{d}r}{[g^2(x)-r^2]^{1/2}}\,\mathcal{L}(r)v(r,\phi). \tag{1.3.21}$$

Here the following property of the Abel operators was used

$$\int_y^a\frac{\mathrm{d}r}{(r^2-y^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a\frac{tf(t)\,\mathrm{d}t}{(t^2-r^2)^{1/2}}=-\frac{\pi}{2}f(y). \tag{1.3.22}$$

Introduction of a new variable $t=g(x)$, $x=l_1(t)$, transforms (21) into

$$V(\rho,\phi,z)=\frac{2}{\pi}\int_0^a\frac{\mathrm{d}l_1(t)}{[\rho^2-l_1^2(t)]^{1/2}}\,\mathcal{L}\!\left(\frac{l_1^2(t)}{\rho t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{\rho_0\mathrm{d}\rho_0}{(t^2-\rho_0^2)^{1/2}}\,\mathcal{L}(\rho_0)v(\rho_0,\phi). \tag{1.3.23}$$

By changing the order of integration in (23), according to the rule

$$\int_0^a F(r)\mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r\frac{\rho f(\rho)\,\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}}=-\int_0^a f(\rho)\mathrm{d}\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a\frac{F(r)r\mathrm{d}r}{(r^2-\rho^2)^{1/2}}, \tag{1.3.24}$$

the following expression can be obtained

$$V(\rho,\phi,z)=-\frac{2}{\pi}\int_0^a\left\{\mathcal{L}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a\frac{t\mathrm{d}l_1(t)}{(t^2-\rho_0^2)^{1/2}[\rho^2-l_1^2(t)]^{1/2}}\,\mathcal{L}\!\left(\frac{\rho}{l_2^2(t)}\right)\right\}v(\rho_0,\phi)\mathrm{d}\rho_0. \tag{1.3.25}$$

The integral in curly brackets can be evaluated in closed form. Consider the following equivalent integral

$$I_1 = -\frac{1}{\rho_0}\,\mathcal{L}(\rho_0)\,\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a}\frac{x\,\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\,\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\,\lambda\!\left(\frac{\rho}{l_2^2(x)},\,\phi-\phi_0\right). \tag{1.3.26}$$

Make use of the rule of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^a\frac{F(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-x^2}} = -\frac{F(a)x}{a(a^2-x^2)^{1/2}} + x\int_x^a\frac{\mathrm{d}\rho}{\sqrt{\rho^2-x^2}}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{F(\rho)}{\rho}\right]$$

$$= -\frac{F(a)a}{x(a^2-x^2)^{1/2}} + \frac{1}{x}\int_x^a\frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-x^2}}\,\frac{\mathrm{d}}{\mathrm{d}\rho}F(\rho). \tag{1.3.27}$$

Expression (26) will take the form

$$I_1 = \frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho_0^2}\,(l_2^2-l_1^2)}\,\lambda\!\left(\frac{\rho\rho_0}{l_2^2},\,\phi-\phi_0\right) - \int_{\rho_0}^{a}\frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\,\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\,\lambda\!\left(\frac{\rho\rho_0}{l_2^2(x)},\,\phi-\phi_0\right)\right]. \tag{1.3.28}$$

Introduce a new variable

$$h(x) = \frac{\sqrt{x^2-\rho_0^2}\,[x^2-l_1^2(x)]^{1/2}}{x},$$

$$h'(x) = \frac{\mathrm{d}h(x)}{\mathrm{d}x} = \frac{[x^2-l_1^2(x)]^{1/2}[x^2 l_2^2(x)-\rho_0^2 l_1^2(x)]}{x^2\sqrt{x^2-\rho_0^2}\,[l_2^2(x)-l_1^2(x)]}\,. \tag{1.3.29}$$

The expression for λ can be presented as

$$\lambda\!\left(\frac{\rho\rho_0}{l_2^2(x)},\,\phi-\phi_0\right) = \frac{[l_2^2(x)-l_1^2(x)]\sqrt{x^2-\rho_0^2}}{[x^2-l_1^2(x)]^{1/2}}\,\frac{h'(x)}{R_0^2+h^2(x)}. \tag{1.3.30}$$

Substitution of (29) and (30) in (28) yields

$$I_1 = \frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho_0^2}\,(l_2^2-l_1^2)}\,\lambda\!\left(\frac{\rho\rho_0}{l_2^2},\,\phi-\phi_0\right)$$

$$-\int_{\rho_0}^{a} \frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xz\sqrt{x^2-\rho_0^2}\,h'(x)}{[x^2-l_1^2(x)][R_0^2+h^2(x)]}\right]$$

$$= \frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho_0^2}(l_2^2-l_1^2)}\,\lambda\left(\frac{\rho\rho_0}{l_2^2}, \phi-\phi_0\right) - z\int_{\rho_0}^{a} \frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(x^2-\rho_0^2)^{3/2}h'(x)}{xh^2(x)[R_0^2+h^2(x)]}\right]. \tag{1.3.31}$$

Integration by parts in (31) yields

$$-\frac{1}{\rho_0}\mathcal{L}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a} \frac{x\mathrm{d}x}{\sqrt{x^2-\rho_0^2}} \frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\,\lambda\left(\frac{\rho}{l_2^2(x)}, \phi-\phi_0\right) = \frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right]. \tag{1.3.32}$$

Here h stands for $h(a)$, as defined by the first expression of (29). In the limiting case, when $a \to \infty$, expression (32) gives yet another representation for z/R_0^3, namely,

$$\frac{z}{R_0^3} = -\frac{2}{\pi\rho_0}\mathcal{L}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{\infty} \frac{x\mathrm{d}x}{\sqrt{x^2-\rho_0^2}} \frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\,\lambda\left(\frac{\rho}{l_2^2(x)}, \phi-\phi_0\right). \tag{1.3.33}$$

Now substitution of (32) in (25) yields

$$V(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right]\frac{z}{R_0^3}\,v(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{1.3.34}$$

Here R_0 is defined by (5) and

$$h = (a^2-l_1^2)^{1/2}(a^2-\rho_0^2)^{1/2}/a. \tag{1.3.35}$$

Formulae (23) and (34) define the potential function V in the half-space $z\geq 0$, expressed directly through its value v prescribed inside the disc $\rho=a$, $z=0$. Expression (23) is useful when an explicit evaluation of the integrals is possible, while expression (34) is more convenient for numerical integration.

Note that in the limiting case, when $z=0$, equation (34) transforms into a known result, namely,

$$V(\rho,\phi,0)=v(\rho,\phi), \quad \text{for} \quad \rho\le a; \quad \text{and}$$

$$V(\rho,\phi,0)=\frac{(\rho^2-a^2)^{1/2}}{\pi^2}\int_0^{2\pi}\int_0^a \frac{v(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]}, \quad \text{for} \quad \rho>a.$$

The solution of the first mixed boundary value problem is completed.

Problem 2. Consider now another internal problem, characterized by the following mixed conditions on the boundary $z=0$:

$$\frac{\partial V}{\partial z}=-2\pi\sigma(\rho,\phi), \quad \text{for}\ \rho\le a, \quad \text{and} \quad 0\le\phi<2\pi;$$

$$V=0, \qquad \text{for}\ \rho>a, \quad \text{and} \quad 0\le\phi<2\pi. \tag{1.3.36}$$

The problem (36) can be interpreted as an electrostatic one of a charged disc $\rho\le a$ inside an infinite grounded diaphragm $\rho>a$. Mathematically similar problem arises in the consideration of a penny-shaped crack subjected to an arbitrary pressure σ.

The potential function V can be represented through the simple layer as follows:

$$V(\rho,\phi,z)=\int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)}{R_0}\rho_0 d\rho_0 d\phi_0+\int_0^{2\pi}\int_a^\infty \frac{\sigma(\rho_0,\phi_0)}{R_0}\rho_0 d\rho_0 d\phi_0. \tag{1.3.37}$$

Substitution of (1.2.13) and (1.2.17) in (37) yields, after interchanging the order of integration

$$V(\rho,\phi,z)=4\int_0^{l_1}\frac{dx}{\sqrt{\rho^2-x^2}}\int_{g(x)}^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+4\int_{l_2}^\infty \frac{dx}{\sqrt{x^2-\rho^2}}\int_a^{g(x)}\frac{\rho_0 d\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.3.38}$$

Here the $\mathcal{L}$-operator is defined by (1.1.4), g is given by (1.2.20), the abbreviations l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively; and the following rule is used for changing the order of integration:

$$\int_0^a \mathrm{d}\rho_0 \int_0^{l_1(\rho_0)} \mathrm{d}x = \int_0^{l_1} \mathrm{d}x \int_{g(x)}^{a} \mathrm{d}\rho_0, \quad \int_a^\infty \mathrm{d}\rho_0 \int_{l_2(\rho_0)}^{\infty} \mathrm{d}x = \int_{l_2}^{\infty} \mathrm{d}x \int_a^{g(x)} \mathrm{d}\rho_0. \tag{1.3.39}$$

Substitution of (36) in (38) leads to the integral equation, for $\rho > a$,

$$\int_0^a \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi)$$

$$+ \int_\rho^\infty \frac{\mathrm{d}x}{\sqrt{x^2 - \rho^2}} \int_a^x \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi) = 0. \tag{1.3.40}$$

Notice that σ in the first term of (40) is known from (36), while σ in the second term is yet to be determined. By using the integral representations (1.2.23) and (1.2.22), equation (40) can be rewritten as

$$\int_\rho^\infty \frac{\mathrm{d}x}{\sqrt{x^2 - \rho^2}} \int_a^x \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi)$$

$$= -\int_\rho^\infty \frac{\mathrm{d}x}{\sqrt{x^2 - \rho^2}} \int_0^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi). \tag{1.3.41}$$

Operation on both sides of (41) by

$$\mathcal{L}(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_t^\infty \frac{\rho \mathrm{d}\rho}{(\rho^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho}\right)$$

leads to

$$\int_a^t \frac{\rho_0 \mathrm{d}\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right) \sigma(\rho_0,\phi) = -\int_0^a \frac{\rho_0 \mathrm{d}\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right) \sigma(\rho_0,\phi). \tag{1.3.42}$$

The next operator to apply is

$$\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{t\mathrm{d}t}{(\rho^2-t^2)^{1/2}}\,\mathcal{L}(t),$$

and the final result takes the form

$$\sigma(\rho,\phi) = -\frac{2}{\pi(\rho^2-a^2)^{1/2}}\int_0^a \frac{\sqrt{a^2-\rho_0^2}\,\rho_0\mathrm{d}\rho_0}{\rho^2-\rho_0^2}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2(\rho^2-a^2)^{1/2}}\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2-\rho_0^2}\,\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \qquad (1.3.43)$$

Formula (43) defines the value of σ outside the circle $\rho = a$ directly through its value inside. Now σ is known all over the plane $z = 0$, and substitution of (43) in the second term of (38) allows us to express the potential function V directly through the prescribed value of σ. The first integration yields

$$V(\rho,\phi,z) = 4\int_0^{l_1} \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_{g(x)}^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$-4\int_{l_2}^\infty \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_0^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \qquad (1.3.44)$$

Here the following integral was employed

$$\int_a^\rho \frac{y\mathrm{d}y}{(\rho^2-y^2)^{1/2}(y^2-a^2)^{1/2}(y^2-r^2)} = \frac{\pi}{2(\rho^2-r^2)^{1/2}(a^2-r^2)^{1/2}}, \quad \text{for } r < a. \qquad (1.3.45)$$

The first term in (44) can be transformed by using (1.2.17), in the following manner:

$$\int_0^{l_1} \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_{g(x)}^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= \int_0^a \rho_0 \mathrm{d}\rho_0 \int_{l_2(\rho_0)}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$= \int_{l_2(0)}^{l_2} \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}} \int_0^{g(x)} \frac{\rho_0\mathrm{d}\rho_0}{[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0\phi)$$

$$+ \int_{l_2}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}} \int_0^{a} \frac{\rho_0\mathrm{d}\rho_0}{[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0\phi). \qquad (1.3.46)$$

Substitution of (46) in (44) yields

$$V(\rho,\phi,z) = 4\int_{l_2(0)}^{l_2} \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}} \int_0^{g(x)} \frac{\rho_0\mathrm{d}\rho_0}{[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0\phi). \qquad (1.3.47)$$

Introduction of a new variable $t = g(x)$, $x = l_2(t)$, transforms (47) into

$$V(\rho,\phi,z) = 4\int_0^{a} \frac{\mathrm{d}l_2(t)}{[l_2^2(t)-\rho^2]^{1/2}} \int_0^{t} \frac{\rho_0\mathrm{d}\rho_0}{(t^2-\rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{l_2^2(t)}\right)\sigma(\rho_0,\phi). \qquad (1.3.48)$$

An interchange of the order of integration in (48), and integration with respect to t (see 1.2.15), yields

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{a} \frac{1}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right)\sigma(\rho_0,\phi_0)\, \rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \qquad (1.3.49)$$

Formulae (47–49) give three equivalent representations of the potential function V, the first two being more convenient for explicit evaluation of the integrals involved, while the third one has some advantages for numerical integration. Two examples are considered below.

Example 1. Let the potential prescribed inside the disc be $v(\rho,\phi) = v_n\rho^n\cos n\phi$, $v_n = \text{const}$. The solution due to (1.3.21) is

$$V(\rho,\phi,z) = \frac{2v_n}{\sqrt{\pi}\,\rho^n}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\cos n\phi\int_0^{l_1}\frac{x^{2n}\mathrm{d}x}{\sqrt{\rho^2-x^2}}$$

$$= v_n\rho^n\cos n\phi\left[1-\frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+\frac{1}{2})}\frac{\sqrt{l_2^2-a^2}}{l_2}F\left(\frac{1}{2}-n,\frac{1}{2};\frac{3}{2};\frac{l_2^2-a^2}{l_2^2}\right)\right].$$

(1.3.50)

The hypergeometric function in (50) can be expressed in elementary functions (Bateman and Erdelyi, 1955)

$$F(\frac{1}{2}-n,\frac{1}{2};\frac{3}{2};\zeta) = \frac{(1-\zeta)^{n+1/2}}{\Gamma(n+1)}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[\frac{\zeta^{n-1/2}}{\sqrt{1-\zeta}}\sin^{-1}\sqrt{\zeta}\right]. \quad (1.3.51)$$

Example 2. Let the charge distribution be prescribed in the form $\sigma(\rho,\phi) = \sigma_n\rho^n\cos n\phi$, $\sigma_n = \text{const}$. The solution is given by (48)

$$V(\rho,\phi,z) = 2\sqrt{\pi}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\sigma_n\rho^n\cos n\phi\int_0^b\frac{x^{2n+2}\mathrm{d}x}{(x^2+z^2)^{n+1}}$$

$$= \sqrt{\pi}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{5}{2})}\sigma_n\rho^n\cos n\phi\,\frac{b^{2n+3}}{z^{2n+2}}F(n+1,n+\frac{3}{2};n+\frac{5}{2};-\frac{b^2}{z^2}),$$

(1.3.52)

where $b = \sqrt{a^2-l_1^2}$, and the hypergeometric function can be expressed in elementary (Bateman and Erdelyi, 1955)

$$F(n+1,n+\frac{3}{2};n+\frac{5}{2};\zeta) = \frac{2n+3}{\Gamma(n+1)}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left\{\frac{1}{\zeta}\left[\frac{1}{2\sqrt{\zeta}}\ln\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}-1\right]\right\}. \quad (1.3.53)$$

1.4. External mixed boundary value problem

The problem is called external when the non-zero boundary conditions are prescribed outside the disc. As in the previous section, we consider two types of problem.

Problem 1. It is necessary to find a function, harmonic in the half-space $z \geq 0$, vanishing at infinity, and subject to the mixed boundary conditions on the plane $z = 0$, namely,

$$\frac{\partial V}{\partial z}\Big|_{z=0} = 0, \quad \text{for } \rho < a,\ 0 \le \phi < 2\pi;$$

$$V = v(\rho,\phi), \quad \text{for } \rho \ge a,\ 0 \le \phi < 2\pi. \qquad (1.4.1)$$

The problem (1) can be interpreted as an electrostatic one of a charged diaphragm, or as an external elastic contact problem. The potential V is presented through a simple layer distribution (1.3.38). Substitution of the boundary conditions (1) in (1.3.38) leads to the governing integral equation

$$4\int_\rho^\infty \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}} \int_a^x \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi) = v(\rho,\phi). \qquad (1.4.2)$$

Its solution is obtained in exactly the same manner as that of (1.3.12), and is

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{x\mathrm{d}x}{\sqrt{\rho^2-x^2}}\mathcal{L}(x^2)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \qquad (1.4.3)$$

The rules of differentiation allow us to rewrite (3) as follows

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{(\rho^2-a^2)^{1/2}} + \int_a^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\frac{\partial}{\partial x}\chi(x,\rho,\phi)\right\}, \qquad (1.4.4)$$

where

$$\chi(x,\rho,\phi) = x\int_x^\infty \frac{\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\frac{\partial}{\partial\rho_0}\left[\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)v(\rho_0\phi)\right]. \qquad (1.4.5)$$

The following transformation can now be performed:

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \frac{\partial}{\partial x}\left[x\int_x^\infty \frac{\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}(\mathcal{L}v)'\right]$$

$$= \int_x^\infty \frac{\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}[(\mathcal{L}v)' + \rho_0(\mathcal{L}v)'' - 2(\mathcal{L}'\rho_0 v)']$$

$$= \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \left[\mathcal{L}\left(v'' + \frac{1}{\rho_0} v' \right) - \left(\mathcal{L}'' + \frac{1}{\rho_0}\mathcal{L}' \right) v \right]. \tag{1.4.6}$$

Here, for the sake of brevity, the primes (′) indicate the partial derivatives with respect to ρ_0, $\mathcal{L}$ stands for $\mathcal{L}(x^2/\rho\rho_0)$, $v \equiv v(\rho_0,\phi)$, and the following identity was used

$$\frac{\partial}{\partial x}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) = -2\left(\frac{\rho_0}{x}\right)\frac{\partial}{\partial \rho_0}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right).$$

Since

$$\mathcal{L}\frac{1}{\rho_0^2}\frac{\partial^2 v}{\partial \phi^2} = \frac{1}{\rho_0^2}\frac{\partial^2 \mathcal{L}}{\partial \phi^2} v,$$

its addition to and subtraction from (6) yields

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \left[\mathcal{L}\Delta v - (\Delta \mathcal{L}) v \right], \tag{1.4.7}$$

where Δ is the two-dimensional Laplace operator in polar coordinates. Since λ is a harmonic function, $\Delta\mathcal{L} = 0$, and (7) simplifies to

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\Delta v. \tag{1.4.8}$$

Substitution of (8) in (4) yields

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{ \frac{\chi(a,\rho,\phi)}{(\rho^2 - a^2)^{1/2}} + \int_a^\rho \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \Delta v(\rho_0,\phi) \right\}. \tag{1.4.9}$$

It should be noticed that the first term in (9) becomes singular when $\rho \to a$, while the second term vanishes at the edge of the disc. In the case of v being a harmonic function, the second term in (105) vanishes, and the solution is represented by the first term only. Further integration with respect to x becomes possible in (9), after interchanging the order of integration, with the result

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{(\rho^2-a^2)^{1/2}}\right.$$

$$\left.+\frac{1}{2\pi}\int_0^{2\pi}\int_a^{\infty}\frac{\Delta v(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}\tan^{-1}\frac{(\rho^2-a^2)^{1/2}(\rho_0^2-a^2)^{1/2}}{a[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}\right\}. \quad (1.4.10)$$

Solutions, like (3) and (9), are appropriate for use when an exact evaluation of the integrals is possible, while the solution in the form (10) has some advantages when numerical integration is to be employed.

Now we can express the potential function V directly through its boundary value v. Since $\sigma = 0$ inside the circle $\rho = a$, the potential function (1.3.38) takes the form

$$V(\rho,\phi,z) = 4\int_{l_2}^{\infty}\frac{dx}{\sqrt{x^2-\rho^2}}\int_a^{g(x)}\frac{\rho_0 d\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \quad (1.4.11)$$

Substitution of (3) in (11) yields, after the first integration,

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_{l_2}^{\infty}\frac{dx}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{\rho g^2(x)}{x^2}\right)\frac{\partial}{\partial g(x)}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \quad (1.4.12)$$

Here the properties of the $\mathcal{L}$-operators (1.1.5) were used, along with the following identity, valid for the Abel-type operators

$$\int_a^{\rho}\frac{dx}{\sqrt{\rho^2-x^2}}\frac{d}{dx}\int_a^x\frac{f(t)\,t dt}{(x^2-t^2)^{1/2}} = \frac{\pi}{2}f(\rho). \quad (1.4.13)$$

Introduction of a new variable $y = g(x)$, $x = l_2(y)$, in (12), allows us to rewrite (12)

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^{\infty}\frac{dl_2(y)}{[l_2^2(y)-\rho^2]^{1/2}}\mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right)\frac{d}{dy}\int_y^{\infty}\frac{\rho_0 d\rho_0}{(\rho_0^2-y^2)^{1/2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \quad (1.4.14)$$

Interchange of the order of integration in (14) yields

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^\infty \left\{ \mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_a^{\rho_0} \frac{y\mathrm{d}l_2(y)}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - \rho^2]^{1/2}}\, \mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right)\right\} v(\rho_0,\phi)\mathrm{d}\rho_0. \tag{1.4.15}$$

Here the general formula was used

$$\int_a^\infty F(\rho)\mathrm{d}\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{xf(x)\mathrm{d}x}{\sqrt{x^2-\rho^2}} = -\int_a^\infty f(x)\,\mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{\rho F(\rho)\,\mathrm{d}\rho}{\sqrt{x^2-\rho^2}}\,. \tag{1.4.16}$$

The integral in curly brackets of (15) can be evaluated in a closed form. We consider the following equivalent integral

$$I_2 = \frac{1}{\rho_0}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_a^{\rho_0} \frac{x\mathrm{d}x}{(\rho_0^2 - x^2)^{1/2}}\, \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\left(\frac{l_1(x)x}{l_2(x)}\,,\phi-\phi_0\right). \tag{1.4.17}$$

Make use of the rule of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{F(\rho)\,\mathrm{d}\rho}{\sqrt{x^2-\rho^2}} = \frac{F(a)x}{a(x^2-a^2)^{1/2}} + x\int_a^x \frac{\mathrm{d}\rho}{\sqrt{x^2-\rho^2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{F(\rho)}{\rho}\right]$$

$$= \frac{F(a)a}{x(x^2-a^2)^{1/2}} + \frac{1}{x}\int_a^x \frac{\rho\mathrm{d}\rho}{\sqrt{x^2-\rho^2}}\frac{\mathrm{d}}{\mathrm{d}\rho}F(\rho). \tag{1.4.18}$$

Expression (17) will take the form

$$I_2 = \frac{\sqrt{a^2 - l_1^2}}{\sqrt{\rho_0^2 - a^2}[l_2^2 - l_1^2]}\,\lambda\left(\frac{l_1^2}{\rho\rho_0}\,,\phi-\phi_0\right)$$

$$+\int_a^{\rho_0} \frac{\mathrm{d}x}{(\rho_0^2 - x^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\left(\frac{l_1(x)x}{l_2(x)\rho_0}\,,\phi-\phi_0\right)\right]. \tag{1.4.19}$$

By introducing the notation

$$F(y) = \frac{z}{R_0^3}\left[\frac{R_0}{j(y)} + \tan^{-1}\left(\frac{j(y)}{R_0}\right)\right], \tag{1.4.20}$$

where $j(x)$ is defined by

$$j(x) = \frac{(\rho_0^2 - x^2)^{1/2}\sqrt{l_2^2(x) - x^2}}{x},$$

$$j'(x) = \frac{\mathrm{d}j(x)}{\mathrm{d}x} = -\frac{\sqrt{l_2^2(x) - x^2}\,[\rho_0^2 l_2^2(x) - x^2 l_1^2(x)]}{x^2[l_2^2(x) - l_1^2(x)]\,(\rho_0^2 - x^2)^{1/2}}. \tag{1.4.21}$$

expression (19) can be rewritten as

$$I_2 = \frac{\rho_0^2 - a^2}{a}\frac{\mathrm{d}F(a)}{\mathrm{d}a} + \int_a^{\rho_0} \frac{\mathrm{d}y}{(\rho_0^2 - y^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{(\rho_0^2 - y^2)^{3/2}}{y}\frac{\mathrm{d}F(y)}{\mathrm{d}y}\right]. \tag{1.4.22}$$

Integration in (22) can be performed by parts, with a simple result $F(a)$, which means establishment of another integral representation

$$\frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right] = \frac{1}{\rho_0}\mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_a^{\rho_0} \frac{x\mathrm{d}x}{(\rho_0^2 - x^2)^{1/2}}\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\lambda\left(\frac{l_1(x)x}{l_2(x)}, \phi - \phi_0\right). \tag{1.4.23}$$

As before, we use the convention $j \equiv j(a)$. Utilization of (23) allows us to simplify (15) as

$$V(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} \frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right] v(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{1.4.24}$$

In the particular case, when $z = 0$, expression (24) simplifies to

$$V(\rho,\phi,0) = \frac{1}{\pi^2}\sqrt{a^2 - \rho^2}\int_0^{2\pi}\int_a^{\infty} \frac{v(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(\rho_0^2 - a^2)^{1/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]}, \quad \text{for } \rho < a;$$

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for } \rho \geq a. \tag{1.4.25}$$

The general solution is completed. The charge density σ is given by the two equivalent expressions (3) and (10), while the potential is in the two forms (14) and (24), the first one being more convenient for exact evaluation of the integrals involved, while the second is better suited for numerical integration.

Problem 2. Consider the problem of finding a harmonic function, vanishing at infinity, and subject to the mixed conditions on the plane $z = 0$

$$V = 0, \quad \text{for } \rho \le a, \;\; 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma(\rho,\phi), \;\text{for } \rho > a, \;\; 0 \le \phi < 2\pi. \tag{1.4.26}$$

The problem may be interpreted as an electrostatic one of a charged infinite diaphragm, with a grounded disc inside, or as an external crack problem in elasticity. Substitution of the boundary conditions (26) in (1.3.38) leads to the governing integral equation

$$\int_0^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= -\int_a^\infty \frac{\mathrm{d}x}{\sqrt{x^2 - \rho^2}} \int_a^x \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.4.27}$$

One should notice that σ in the second term of (27) is known from the boundary condition (26), while the value of σ in the first term is yet to be determined. The right hand side of (27) can be transformed, by using (1.2.22) and (1.2.23),

$$\int_0^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= -\int_0^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_a^\infty \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi),$$

with an immediate result

$$\int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi) = -\int_a^\infty \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi). \tag{1.4.28}$$

Application of the operator

$$\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{x\mathrm{d}x}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{1}{x}\right)$$

to both sides of (28) gives, after necessary transformations

$$\sigma(\rho,\phi) = -\frac{2}{\pi\sqrt{a^2-\rho^2}}\int_a^\infty \frac{(\rho_0^2-a^2)^{1/2}}{\rho_0^2-\rho^2}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma(\rho_0,\phi)\,\rho_0\mathrm{d}\rho_0, \quad \text{for } \rho<a, \tag{1.4.29}$$

or, interpreting the $\mathcal{L}$-operator, we obtain

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\sqrt{a^2-\rho^2}}\int_0^{2\pi}\int_a^\infty \frac{(\rho_0^2-a^2)^{1/2}\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{1.4.30}$$

Now the value of σ is known all over the plane $z=0$, and (1.3.38) can be used in order to express the potential V directly through the prescribed σ. Substitution of (29) in (1.3.38) yields, after the first integration

$$V(\rho,\phi,z) = -4\int_0^{l_1}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_a^\infty \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+4\int_{l_2}^\infty\frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_a^{g(x)} \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.4.31}$$

The second term in (31) is equivalent to the second term in (1.3.38), which, in turn, can be represented by using (1.2.13), as

$$4\int_a^\infty\left\{\int_0^{l_1(\rho_0)}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-g^2(x)}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\right\}\sigma(\rho_0,\phi)\,\rho_0\mathrm{d}\rho_0.$$

The following scheme of changing the order of integration is enacted

$$\int_a^\infty \mathrm{d}\rho_0\int_0^{l_1(\rho_0)}\mathrm{d}x = \int_0^{l_1}\mathrm{d}x\int_a^\infty \mathrm{d}\rho_0 + \int_{l_1}^{l_1(\infty)}\mathrm{d}x\int_{g(x)}^\infty \mathrm{d}\rho_0,$$

and the second term in (31) can be rewritten as

$$4\int_0^{l_1}\frac{dx}{\sqrt{\rho^2-x^2}}\int_a^{\infty}\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+\,4\int_{l_1}^{l_1(\infty)}\frac{dx}{\sqrt{\rho^2-x^2}}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \qquad (1.4.32)$$

Substitution of (32) in (31) gives, by virtue of $l_1(\infty) = \rho$,

$$V(\rho,\phi,z) = 4\int_{l_1}^{\rho}\frac{dx}{\sqrt{\rho^2-x^2}}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \qquad (1.4.33)$$

Interchange of the order of integration in (33), and integration with respect to x, according to (1.2.19), results in

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty}\frac{1}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0, \qquad (1.4.34)$$

where R_0 is defined by (1.3.5), and j stands for $j(a)$, as defined by (21).

The second problem is now solved. Expression (30) defines the charge density σ inside a circle directly in terms of its values outside. The potential V is given by two equivalent expressions (33) and (34), the first one to be used for exact evaluation of the integrals, while the second has some advantages in the case of numerical integration. Some specific examples are considered below.

Example 1. Consider an external mixed problem with the following boundary conditions at $z = 0$

$$V = v_0/\rho^n, \quad \text{for } \rho \ge a, \quad 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, \quad 0 \le \phi < 2\pi. \qquad (1.4.35)$$

The conditions (35) correspond to those of Problem 1. The solution is given by (14) and (3). Substitution of (35) in (14) yields, after the integration

$$V(\rho,\phi,z) = \frac{2v_0}{\sqrt{\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)} \int_{l_2}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}\, g^n(x)}, \tag{1.4.36}$$

where $g(x)$ is defined by (1.2.20), and the following integral was employed (Gradshtein and Ryzhik, 1963)

$$\int_x^{\infty} \frac{\mathrm{d}\rho}{\rho^n \sqrt{\rho^2-x^2}} = \frac{\sqrt{\pi}\,\Gamma(n/2)}{2\Gamma[(n+1)/2]\, x^n}. \tag{1.4.37}$$

The integral in (36) can be evaluated in terms of elementary functions for any integer n, but the procedure is slightly different for even and odd values of n. For example, for even $n = 2k$, the problem reduces to the evaluation of the integral

$$\int_{l_2}^{\infty} \frac{(x^2-\rho^2)^{k-1/2}\mathrm{d}x}{x^{2k}(x^2-\rho^2-z^2)^k},$$

which can be evaluated by introduction of a new variable $t = x/\sqrt{x^2 - \rho^2}$. The final result is

$$V(\rho,\phi,z) = \frac{2v_0}{\sqrt{\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)z^n} \Bigg\{ \sum_{m=1}^{k} \frac{A_m}{2m-1} [1-Q_0^{2m-1}]$$

$$+ 2B_1 \ln Q + \sum_{m=2}^{k} \frac{B_m}{1-m} \left[(Q_1^{m-1}-Q_2^{m-1}) - (Q_3^{m-1}-Q_4^{m-1}) \right] \Bigg\}, \tag{1.4.38}$$

where

$$A_{k-m+1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\eta^{m-1}} \left[\frac{(\eta-1)^{k-1}}{(r^2-\eta)^k} \right], \text{ for } \eta = 0, \text{ and } r^2 = 1+\rho^2/z^2;$$

$$B_{k-m+1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{(t^2-1)^{k-1}}{t^{2k}(r^2+t)^k} \right], \text{ for } t = \sqrt{1+\rho^2/z^2};$$

$$Q_0 = \frac{\sqrt{l_2^2-\rho^2}}{l_2}, \quad Q = \frac{l_2[(\rho^2+z^2)^{1/2} + \sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]},$$

$$Q_1 = \frac{z[(\rho^2+z^2)^{1/2} + \sqrt{l_2^2 - a^2}]}{l_1^2}, \quad Q_2 = \frac{z[(\rho^2+z^2)^{1/2} - \sqrt{l_2^2 - a^2}]}{l_1^2},$$

$$Q_3 = \frac{z[(\rho^2+z^2)^{1/2} + z]}{\rho^2}, \quad Q_4 = \frac{z[(\rho^2+z^2)^{1/2} - z]}{\rho^2}. \tag{1.4.39}$$

For the case of an odd $n = 2k+1$, the integration can be performed by using the substitution $t = (x^2 - \rho^2 - z^2)^{1/2}$, and the final result is

$$V(\rho,\phi,z) = \frac{v_0}{\sqrt{\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)} \left\{ \sum_{m=1}^{k} \frac{C_m}{(2m-1)(a^2 - l_1^2)^{m-1/2}} + \sum_{m=1}^{k+1} D_m E_m \right\}, \tag{1.4.40}$$

where

$$C_m = \frac{1}{(k-m)!} \frac{\mathrm{d}^{k-m}}{\mathrm{d}t^{k-m}} \left[\frac{(t+z^2)^k}{(t+\rho^2+z^2)^{k+1}} \right], \text{ for } t = 0;$$

$$D_m = \frac{1}{(k+1-m)!} \frac{\mathrm{d}^{k+1-m}}{\mathrm{d}t^{k+1-m}} \left[\frac{(t+z^2)^k}{t^k} \right], \text{ for } t = -(\rho^2+z^2);$$

$$E_m = \frac{(-1)^m}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{1}{\sqrt{t}} \tan^{-1} \frac{\sqrt{t}}{\sqrt{a^2 - l_1^2}} \right], \text{ for } t = \rho^2 + z^2. \tag{1.4.41}$$

Substitution of (35) in (3) yields, after integration,

$$\sigma(\rho,\phi) = \frac{v_0 \Gamma[(n+1)/2]}{\pi^{3/2}\Gamma(n/2)} \left\{ \frac{1}{a^n(\rho^2 - a^2)^{1/2}} - \frac{n(\rho^2-a^2)^{1/2}}{\rho^{n+2}} F\left(\tfrac{1}{2}n+1, \tfrac{1}{2}; \tfrac{3}{2}; 1 - \frac{a^2}{\rho^2} \right) \right\}, \tag{1.4.42}$$

and the Gauss hypergeometric function can be expressed in elementary functions (Bateman and Erdelyi, 1955), namely, for even $n = 2k$, $k = 1,2,3\ \ldots$,

$$F\left(k+1, \frac{1}{2}; \frac{3}{2}; t\right) = \frac{1}{2k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[t^{k-1/2} \ln \frac{1+\sqrt{t}}{1-\sqrt{t}} \right],$$

and for odd $n = 2k+1$, $k = 0,1,2,\ \ldots$,

$$F\left(k+\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; t\right) = \frac{\sqrt{\pi}}{2\sqrt{t}\,\Gamma(k+\frac{3}{2})} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[\frac{t^{k+1/2}}{\sqrt{1-t}} \right]. \tag{1.4.43}$$

Fig. 1.4.1. Charge density for $n = 1, 2, 3, 4$.

The dimensionless charge density distribution $\sigma^* = \sigma a^{n+1}/v_0$, evaluated due to (42) for $n = 1, 2, 3, 4$, is presented on Fig. 1.4.1 versus $\rho^* = \rho/a$. It is non-negative for $n = 1$, and changes sign when $n \geq 2$, its negative maximum increases with n, while the total charge stays at zero. Some specific formulae may be found in Exercises 1 (Examples 23-26). The equipotential lines for $n = 2$ (formula 40) are presented in Fig. 1.4.2.

Fig. 1.4.2. Equipotential lines for $n = 2$

Example 2. Consider the boundary conditions at $z = 0$

$$V = (v_n/\rho^n)\, e^{in\phi}, \quad \text{for } \rho \geq a, \; 0 \leq \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, \; 0 \leq \phi < 2\pi, \tag{1.4.44}$$

where v_n is constant. The solution is given by (3) and (14). Substitution of (44) in (14) yields, after integration,

$$V(\rho,\phi,z) = \frac{2\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n)}\, \rho^n e^{in\phi} \int_{l_2}^{\infty} \frac{dx}{x^{2n}\sqrt{x^2-\rho^2}}\,. \tag{1.4.45}$$

The final integration gives

$$V(\rho,\phi,z) = \frac{2v_n}{\sqrt{\pi}\rho^n} e^{in\phi} \sum_{k=1}^{n} \frac{(-1)^{k-1}\Gamma(n+1/2)}{(2k-1)\Gamma(k)\Gamma(n+1-k)} (1 - Q_0^{2k-1}), \tag{1.4.46}$$

where Q_0 is defined by (39). Some particular cases of (46) can be found in Exercises 1 (Examples 27-29). Substituting (44) in (3), we get, after integration,

$$\sigma(\rho,\phi) = \frac{\Gamma(n+1/2)}{\pi^{3/2}\Gamma(n)} \frac{v_n e^{in\phi}}{\rho^n \sqrt{\rho^2 - a^2}}. \tag{1.4.47}$$

Evidently, expression (47) can also be obtained by differentiation of (46) with respect to z for $z = 0$. The equipotential lines at the plane $\phi = 0$ for $n = 2$ are presented in Fig. 1.4.3.

Fig. 1.4.3. Equipotential lines for $n = 2$

Example 3. Consider a case related to Problem 2, with the boundary conditions

$$V = 0, \quad \text{for } \rho \le a, \ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi \frac{\sigma_0}{\rho^n} \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi; \tag{1.4.48}$$

The solution is given by (29) and (33). Substitution of (48) in (33) yields, after integration using (37),

$$V(\rho,\phi,z) = 2\sqrt{\pi}\sigma_0 \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \int_{l_1}^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}\, g^{n-1}(x)}, \tag{1.4.49}$$

where $g(x)$ is defined by (1.2.20). The technique used in the previous example can be employed here for further integration. The final result depends on the value of n being even or odd. For even $n = 2k$, $k = 1,2,3,\ \ldots,$ the potential is

$$V(\rho,\phi,z) = 2\sqrt{\pi}\sigma_0 \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \Bigg\{ 2B_1 \ln Q$$

$$+ \sum_{m=1}^{k-1} \frac{A_m}{(2m-1)\, z^{2m-1}} \left[1 - \left(\frac{\sqrt{a^2 - l_1^2}}{a} \right)^{2m-1} \right]$$

$$+ \sum_{m=1}^{k-1} \frac{B_{m+1}}{m z^m} [Q_1^m - Q_2^m - (Q_3^m - Q_4^m)] \Bigg\}. \tag{1.4.50}$$

Here Q, Q_1, Q_2, Q_3, and Q_4 are defined by (39), and

$$A_{k-m} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{(t - z^2)^{k-1}}{(\rho^2 + z^2 - t)^k} \right] \quad \text{for } t=0;$$

$$B_{k-m+1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{(t^2 - z^2)^{k-1}}{t^{2k-2}[(\rho^2 + z^2)^{1/2} - t]^k} \right], \quad \text{for } t = -(\rho^2 + z^2)^{1/2}. \tag{1.4.51}$$

For odd $n = 2k + 1$, the result is

$$V(\rho,\phi,z) = 2\sqrt{\pi}\ \frac{\sigma_0 \Gamma[(n-1)/2]}{\Gamma(n/2)\, z^{n-1}} \sum_{m=1}^{k} \left\{ \frac{G_m}{2m-1} \left(\frac{\sqrt{l_2^2 - a^2}}{a} \right)^{2m-1} - H_m L_m \right\}. \tag{1.4.52}$$

Here

$$G_{k\text{-}m+1} = \frac{1}{\Gamma(m)} \frac{\mathrm{d}^{m\text{-}1}}{\mathrm{d}t^{m\text{-}1}} \left[\frac{(1+t)^{k\text{-}1}}{(\xi+t)^k} \right], \quad \text{for } t = 0,\ \xi = (\rho^2 + z^2)/z^2;$$

$$H_{k\text{-}m+1} = \frac{1}{\Gamma(m)} \frac{\mathrm{d}^{m\text{-}1}}{\mathrm{d}t^{m\text{-}1}} \left[\frac{(1+t)^{k\text{-}1}}{t^k} \right], \quad \text{for } t = -(\rho^2 + z^2)/z^2;$$

$$L_m = \frac{(-1)^m}{\Gamma(m)} \frac{\mathrm{d}^{m\text{-}1}}{\mathrm{d}t^{m\text{-}1}} \left[\frac{1}{\sqrt{t}} \tan^{-1}\left(\frac{\sqrt{t}}{a} \sqrt{l_2^2 - a^2} \right) \right], \quad \text{for } t = (\rho^2 + z^2)/z^2. \qquad (1.4.53)$$

Fig. 1.4.4. Charge distribution for $n = 2, 3, 4$

Some particular cases of (50) and (52) can be found in Exercises 1 (Examples 30-32). The dimensionless charge density distribution $\sigma^* = \sigma a^n/\sigma_0$ is given in Fig. 1.4.4 versus $\rho^* = \rho/a$ for $n = 2, 3, 4$. The equipotential lines for $n = 3$ are presented in Fig. 1.4.5. The dimensionless potential $v^* = Va^2/\sigma_0$ was varied from 0.5 to 1.3. We note that the equipotential lines for $v^* < 0.92$ have two branches.

Example 4. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_n/\rho^n)\mathrm{e}^{in\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi; \qquad (1.4.54)$$

Fig. 1.4.5. Equipotential lines for $n = 3$

Substitution of (54) in (33) yields, after integration,

$$V(\rho,\phi,z) = 2\sqrt{\pi}\ \frac{\sigma_n \Gamma(n-1/2)}{\Gamma(n)\rho^n}\, e^{in\phi} \left\{ \sqrt{l_2^2 - a^2} - z \right.$$

$$\left. - z \sum_{k=2}^{n} \frac{(-1)^k \Gamma(n)}{\Gamma(k)\,\Gamma(n-k+1)(2k-3)} \left[1 - (1 - l_1^2/a^2)^{k-3/2}\right] \right\}, \tag{1.4.55}$$

and on the plane $z = 0$

$$V(\rho,\phi,0) = 2\sqrt{\pi}\ \frac{\sigma_n \Gamma(n-1/2)}{\Gamma(n)\rho^n}\, e^{in\phi} \Re \sqrt{\rho^2 - a^2}.$$

The symbol $\Re$ indicates the real part sign. The charge density is defined, according to (29),

$$\sigma(\rho,\phi) = \frac{\sigma_n \Gamma(n-1/2)}{\sqrt{\pi}\,\Gamma(n)\rho^n}\, e^{in\phi} \Re \left\{ 1 - \frac{a}{\sqrt{a^2 - \rho^2}} \right.$$

$$+\sum_{k=2}^{n}\frac{(-1)^k\Gamma(n)}{\Gamma(k)\,\Gamma(n-k+1)(2k-3)}\,[1-(1-\rho^2/a^2)^{k-3/2}]\Bigg\}. \tag{1.4.56}$$

A more general case of boundary conditions, namely,

$$V=0,\quad \text{for } \rho\le a,\quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=-2\pi(\sigma_{jn}/\rho^j)e^{in\phi},\quad \text{for } \rho>a,\quad 0\le\phi<2\pi; \tag{1.4.57}$$

can also be considered, by using the same technique as in the previous examples, and the final result can always be expressed in elementary functions. The form of the result will be different for $(j+n)$ even, and for $(j+n)$ odd. As an example, the following expression can be obtained by substituting (57) in (29), for the case when $j+n=2k$

$$\sigma(\rho,\phi)=\frac{\sigma_{jn}}{\rho^j}\,e^{in\phi}\Re\left\{1-\frac{a}{\sqrt{a^2-\rho^2}}\left[1-\sum_{m=2}^{k}\frac{\Gamma(m-3/2)}{2\sqrt{\pi}\,\Gamma(m)}\left(\frac{\rho}{a}\right)^{2m-2}\right]\right\}, \tag{1.4.58}$$

and for odd $j+n=2k+1$

$$\sigma(\rho,\phi)=\frac{2\sigma_{jn}}{\pi\rho^j}\,e^{in\phi}\,\Re\left\{\sin^{-1}\!\left(\frac{\rho}{a}\right)-\frac{a}{\sqrt{a^2-\rho^2}}\left[1-\sum_{m=2}^{k}\frac{\sqrt{\pi}\,\Gamma(m-1)}{4\Gamma(m+1/2)}\left(\frac{\rho}{a}\right)^{2m-2}\right]\right\}, \tag{1.4.59}$$

Expressions (58) and (59) represent general formulae which cover all the particular cases considered in Examples 3 and 4.

The examples above have demonstrated the simplicity of the method. The generation of the solution is reduced to a straightforward and elementary procedure.

1.5. Mixed Problems in Spherical Coordinates

Exact solution in closed form is obtained to the following mixed problem for a charged sphere: an arbitrary charge density distribution is prescribed at the surface of a spherical cap while an arbitrary potential is given at the rest of the sphere. The new method makes the solution straightforward and elementary, with no special functions or integral transforms involved. A new type of solution is obtained for the Dirichlet problem with discontinuous boundary conditions.

Integral representation for the reciprocal of the distance between two points in spherical coordinates. Consider two points in spherical coordinates $M(r,\theta,\phi)$ and $N(a,\theta_0,\phi_0)$. The parameters l_1 and l_2 introduced in section 1.2 have the geometrical interpretation as the difference and the sum of the shortest and the longest distance from a point to the edge of a circular disk. In spherical coordinates the same quantities with respect to a spherical cap can be expressed as

$$m_1(\theta_0,\theta,a,r) = \frac{1}{2}\left[\sqrt{a^2+r^2-2ar\cos(\theta+\theta_0)} - \sqrt{a^2+r^2-2ar\cos(\theta-\theta_0)}\right]$$

$$m_2(\theta_0,\theta,a,r) = \frac{1}{2}\left[\sqrt{a^2+r^2-2ar\cos(\theta+\theta_0)} + \sqrt{a^2+r^2-2ar\cos(\theta-\theta_0)}\right] \tag{1.5.1}$$

The following properties can be easily established:

$$m_1 m_2 = ra\sin\theta\sin\theta_0, \quad m_1^{\,2} + m_2^{\,2} = r^2 + a^2 - 2ar\cos\theta\cos\theta_0, \tag{1.5.2}$$

so that the distance between two points M and N can be expressed as $R_0^2 = m_1^{\,2} + m_2^{\,2} - 2m_1m_2\cos(\phi-\phi_0)$. This property allows us to use formulae from section 1.2. For example, we can derive the following integral representations

$$\frac{1}{R_0} = \frac{1}{\pi\sqrt{ar}}\int_0^{t_1(\theta_0)} \frac{\lambda\left(\dfrac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\tau-\cos\theta}\,\sqrt{\cos\gamma(\tau)-\cos\theta_0}}, \tag{1.5.3}$$

$$\frac{1}{R_0} = \frac{1}{\pi\sqrt{ar}}\int_{t_2(\theta_0)}^{\pi} \frac{\lambda\left(\dfrac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\theta-\cos\tau}\,\sqrt{\cos\theta_0-\cos\gamma(\tau)}}, \tag{1.5.4}$$

where

$$t_1 \equiv t_1(\theta_0,\theta,a,r) = 2\tan^{-1}\left(\frac{m_1(\theta_0)}{2\sqrt{ar}\cos(\theta/2)\cos(\theta_0/2)}\right)$$

$$= 2\tan^{-1}\left[\frac{m_1(\theta_0)}{m_2(\theta_0)}\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\right]^{1/2},$$

$$t_2 \equiv t_2(\theta_0,\theta,a,r) = 2\tan^{-1}\left(\frac{m_2(\theta_0)}{2\sqrt{ar}\cos(\theta/2)\cos(\theta_0/2)}\right)$$

$$= 2\tan^{-1}\left[\frac{m_2(\theta_0)}{m_1(\theta_0)}\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\right]^{1/2}, \tag{1.5.5}$$

$$\cos\gamma(\tau) = \cos\tau - \frac{(r-a)^2}{4ar}\frac{\sin^2\tau}{\cos\tau - \cos\theta}, \tag{1.5.6}$$

and hereafter $m_1(x)$ and $m_2(x)$ are understood as abbreviations for $m_1(x, \theta, a, r)$ and $m_2(x, \theta, a, r)$ respectively. It is possible to show that both t_1 and t_2 are inverse to γ, i.e. $\gamma[t_{1,2}(\theta_0] = \theta_0$. Note also that $t_1 \le \min(\theta, \theta_0)$ and $t_2 \ge \max(\theta, \theta_0)$. We can see certain analogy between the notations and their properties used in cylindrical coordinates and those in spherical coordinates: l corresponds to t, g corresponds to γ, etc.

By using analogy with section 1.2, we can derive the following indefinite integrals:

$$\int \frac{\lambda\left(\dfrac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi - \phi_0\right)d\tau}{\sqrt{\cos\tau - \cos\theta}\sqrt{\cos\gamma(\tau) - \cos\theta_0}} = -\frac{2\sqrt{ar}}{R_0}\tan^{-1}\left(\frac{y_1(\tau)}{R_0}\right), \tag{1.5.7}$$

$$\int \frac{\lambda\left(\dfrac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi - \phi_0\right)d\tau}{\sqrt{\cos\theta - \cos\tau}\sqrt{\cos\theta_0 - \cos\gamma(\tau)}} = \frac{2\sqrt{ar}}{R_0}\tan^{-1}\left(\frac{y_2(\tau)}{R_0}\right), \tag{1.5.8}$$

where

$$y_1(\tau) = 2\sqrt{ar}\ \ \sqrt{\cos\tau - \cos\theta}\ \ \sqrt{\cos\gamma(\tau) - \cos\theta_0}/\sin\tau,$$

$$y_2(\tau) = 2\sqrt{ar}\ \ \sqrt{\cos\theta - \cos\tau}\ \ \sqrt{\cos\theta_0 - \cos\gamma(\tau)}/\sin\tau, \tag{1.5.9}$$

$$R_0^2 = r^2 + a^2 - 2ar[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)]. \tag{1.5.10}$$

The integrals in (7) and (8) can be verified by using the identity

$$\lambda\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi - \phi_0\right) = \frac{\sin\tau\ y_1(\tau)}{R_0^2 + y_1(\tau)}\frac{dy_1(\tau)}{d\tau}. \tag{1.5.11}$$

A similar relationship can be established to verify (8).

Formulation of the problem. We consider the following general problem: it is necessary to find the electrostatic field of a charged sphere of radius a when an arbitrary potential v is prescribed over a spherical cap $0 \le \theta \le \alpha$, while an arbitrary charge distribution σ is given at the rest of the sphere. As before, we represent the potential through a simple layer

$$V(r,\theta,\phi)=\int_0^{2\pi}\mathrm{d}\phi_0\int_0^{\alpha}\frac{\sigma(\theta_0,\phi_0)\,a^2\sin\theta_0\,\mathrm{d}\theta_0}{R_0}+\int_0^{2\pi}\mathrm{d}\phi_0\int_{\alpha}^{\pi}\frac{\sigma(\theta_0,\phi_0)\,a^2\sin\theta_0\,\mathrm{d}\theta_0}{R_0},\tag{1.5.12}$$

where R_0 is defined by (10).

Substitution of (3) and (4) in (12) yields, after interchanging the order of integration

$$V(r,\theta,\phi)=\frac{2a^2}{\sqrt{ar}}\left\{\int_0^{t_1(\alpha)}\frac{\mathrm{d}\tau}{\sqrt{\cos\tau-\cos\theta}}\int_{\gamma(\tau)}^{\alpha}\frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos\gamma(\tau)-\cos\theta_0}}\,\mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi)\right.$$

$$\left.+\int_{t_2(\alpha)}^{\pi}\frac{\mathrm{d}\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{\alpha}^{\gamma(\tau)}\frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos\theta_0-\cos\gamma(\tau)}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)\right\}.\tag{1.5.13}$$

It is convenient at this stage to split our problem in two: *i*) to find the electrostatic potential of a charged sphere when an arbitrary charge density is given at a spherical cap, and the zero potential is prescribed elsewhere; *ii*) to find the potential when the zero charge density is prescribed at a spherical cap, and an arbitrary potential is given elsewhere. Both problems are treated separately.

Problem 1. Consider the boundary value problem, with the following mixed conditions at $r=a$:

$$\sigma=\sigma(\theta,\phi),\quad \text{for}\quad 0\le\phi<2\pi,\quad 0\le\theta<\alpha;$$

$$V(a,\theta,\phi)=0,\quad \text{for}\quad 0\le\phi<2\pi,\quad \alpha<\theta\le\pi.\tag{1.5.14}$$

Substitution of the boundary conditions (14) in (13) yields

$$0=\int_0^{\alpha}\frac{\mathrm{d}\tau}{\sqrt{\cos\tau-\cos\theta}}\int_{\tau}^{\alpha}\frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos\tau-\cos\theta_0}}\,\mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi)$$

$$+\int_{\theta}^{\pi}\frac{\mathrm{d}\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{\alpha}^{\tau}\frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi).\tag{1.5.15}$$

Notice that the value of σ in the first term of (15) is known from (14) while σ in the second term of (15) is as yet unknown. It is then necessary to express one through the other. By using (4), we can rewrite (15) in the following manner:

$$\int_\theta^\pi \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_\alpha^\tau \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)$$

$$= -\int_\theta^\pi \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_0^\alpha \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi),$$

which immediately simplifies as

$$\int_\alpha^\tau \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\tau/2)}\right)\sigma(\theta_0,\phi) = -\int_0^\alpha \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\tau/2)}\right)\sigma(\theta_0,\phi). \tag{1.5.16}$$

Application of the operator

$$-\frac{1}{\sin\theta}\mathcal{L}\left(\frac{1}{\tan(\theta/2)}\right)\frac{d}{d\theta}\int_\alpha^\theta \frac{\sin\tau\, d\tau}{\sqrt{\cos\tau - \cos\theta}} \mathcal{L}[\tan(\tau/2)]$$

to both sides of (16) gives

$$\sigma(\theta,\phi) = -\frac{1}{\pi\sqrt{\cos\alpha - \cos\theta}} \int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}\ \sin\theta_0\, d\theta_0}{\cos\theta_0 - \cos\theta} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta/2)}\right)\sigma(\theta_0,\phi). \tag{1.5.17}$$

Expression (17) can also be rewritten in the form

$$\sigma(\theta,\phi) = -\frac{1}{2\pi^2\sqrt{\cos\alpha - \cos\theta}} \int_0^{2\pi}\int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}\ \sigma(\theta_0,\phi_0)\sin\theta_0\, d\theta_0\, d\phi_0}{1 - \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos(\phi - \phi_0)}, \tag{1.5.18}$$

which corresponds to the Green’s function found in a geometric form by Lord Kelvin who used his method of images. Certain simplification occurs in the case of axial symmetry, namely,

$$\sigma(\theta) = -\frac{1}{\pi\sqrt{\cos\alpha - \cos\theta}} \int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}}{\cos\theta_0 - \cos\theta}\, \sigma(\theta_0) \sin\theta_0 \, d\theta_0. \tag{1.5.19}$$

The charge density is now known all over the sphere from (14) and (17). Substitution of (17) in the second term (13) yields, after simplification,

$$V(r,\theta,\phi) = \frac{2a^2}{\sqrt{ar}} \left\{ \int_0^{t_1(\alpha)} \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_{\gamma(\tau)}^\alpha \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}} \right.$$

$$\times \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)} \right) \sigma(\theta_0, \phi)$$

$$\left. + \int_{t_2(\alpha)}^\pi \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_0^\alpha \frac{\sin\theta_1 \, d\theta_1}{\sqrt{\cos\theta_1 - \cos\gamma(\tau)}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_1/2)}{\tan^2(\tau/2)} \right) \sigma(\theta_1, \phi) \right\}. \tag{1.5.20}$$

The first term in (20) can be transformed in the following manner:

$$\int_0^{t_1(\alpha)} \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_{\gamma(\tau)}^\alpha \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}} \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)} \right) \sigma(\theta_0, \phi)$$

$$= \int_0^\alpha \sin\theta_0 \, d\theta_0 \int_{t_2(\theta_0)}^\pi \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}\sqrt{\cos\theta_0 - \cos\gamma(\tau)}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)} \right) \sigma(\theta_0, \phi). \tag{1.5.21}$$

The interchange of the order of integration in (21) can be performed according to the scheme

$$\int_0^\alpha d\theta_0 \int_{t_2(\theta_0)}^\pi d\tau = \int_{t_2(0)}^{t_2(\alpha)} d\tau \int_0^{\gamma(\tau)} d\theta_0 + \int_{t_2(\alpha)}^\pi d\tau \int_0^\alpha d\theta_0, \tag{1.5.22}$$

and the back substitution in (20) yields

$$V(r,\theta,\phi)=\frac{2a^2}{\sqrt{ar}}\int_{t_2(0)}^{t_2(\alpha)}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_0^{\gamma(\tau)}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\gamma(\tau)}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi). \tag{1.5.23}$$

Interchange in the order of integration in (23) and subsequent integration with respect to τ (see (8)) results in

$$V(r,\theta,\phi)=\frac{2}{\pi}\int_0^{2\pi}d\phi_0\int_0^{\alpha}\frac{\sigma(\theta_0,\phi_0)}{R_0}\tan^{-1}\left(\frac{\xi}{R_0}\right)a^2\sin\theta_0\,d\theta_0, \tag{1.5.24}$$

where R_0 is defined by (10) and

$$\xi=\frac{\sqrt{2}\sqrt{\cos\theta_0-\cos\alpha}\sqrt{m_2^2(\alpha)-\cos^2(\alpha/2)\,m_2^2(0)}}{\sin\alpha}. \tag{1.5.25}$$

Notice certain similarity between (25) and (1.3.49). Expressions (23) and (24) give two equivalent solutions to the problem 1, the first one being more convenient for the exact evaluation of the integrals involved while the second one has certain advantages when a numerical integration is required.

Problem 2. Consider a charged sphere with the following boundary conditions at its surface $r=a$

$$\sigma(\theta,\phi)=0,\ \text{for}\ \ 0\le\phi<2\pi,\ \ 0\le\theta<\alpha;$$

$$V(a,\theta,\phi)=v(\theta,\phi),\ \text{for}\ 0\le\phi<2\pi,\ \ \alpha\le\theta\le\pi. \tag{1.5.26}$$

The following integral equation results after substituting (26) in (13).

$$2a\int_\theta^{\pi}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_\alpha^{\tau}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)=v(\theta,\phi). \tag{1.5.27}$$

Application of the operator

$$\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_{\theta_1}^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)$$

to both sides of (27) yields

$$-2\pi a\int_{\alpha}^{\theta_1}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\theta_1}}\,\mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta_1/2)}\right)\sigma(\theta_0,\phi)=\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \qquad (1.5.28)$$

The next operator to apply is

$$\frac{1}{\sin\theta_2}\,\mathcal{L}\left(\cot\frac{\theta_2}{2}\right)\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan\frac{\theta_1}{2}\right),$$

with the final result

$$\sigma(\theta_2,\phi)=-\frac{\mathcal{L}[\cot(\theta_2/2)]}{2\pi^2 a\sin\theta_2}\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \qquad (1.5.29)$$

Now the following rules of differentiation can be used

$$\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2}\frac{f(\theta_1)\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}=\sin\theta_2\left[\frac{f(\alpha)}{\sqrt{\cos\alpha-\cos\theta_2}}\right.$$

$$\left.+\int_{\alpha}^{\theta_2}\frac{df(\theta_1)}{d\theta_1}\frac{d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\right],$$

$$\frac{d}{d\theta_1}\int_{\theta_1}^{\pi}\frac{f(\theta)\sin\theta\, d\theta}{\sqrt{\cos\theta_1-\cos\theta}} = 2\tan\frac{\theta_1}{2}\int_{\theta_1}^{\pi}\frac{\cos(\theta/2)\, d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\frac{d}{d\theta}\left[\cos\left(\frac{\theta}{2}\right)f(\theta)\right]. \tag{1.5.30}$$

By using (7) and (30), expression (29) can be simplified as follows:

$$\sigma(\theta_2,\phi) = -\frac{1}{2\pi^2 a}\left\{\frac{\Phi(\alpha,\theta_2,\phi)}{\sqrt{\cos\alpha-\cos\theta_2}} + \int_{\alpha}^{\theta_2}\frac{d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\frac{\partial}{\partial\theta_1}\Phi(\theta_1,\theta_2,\phi)\right\}. \tag{1.5.31}$$

Here

$$\Phi(\theta_1,\theta_2,\phi) = 2\tan\frac{\theta_1}{2}\int_{\theta_1}^{\pi}\frac{\cos(\theta/2)\, d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\frac{d}{d\theta}\left[\cos\left(\frac{\theta}{2}\right)\mathcal{L}\left(\frac{\tan^2(\theta_1/2)}{\tan(\theta/2)\tan(\theta_2/2)}\right)v(\theta,\phi)\right]. \tag{1.5.32}$$

Formulae (29) and (31)–(32) give two equivalent forms of solution of the integral equation (27). We note two different terms in (31): the first one is singular at $\theta_2\to\alpha$, while the second one tends to zero at $\theta_2\to\alpha$.

The potential in space due to a charged sphere can be obtained by substitution of (29) into (13). The result is

$$V(r,\theta,\phi) = -\frac{\sqrt{a}}{\pi\sqrt{r}}\int_{t_2(\alpha)}^{\pi}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\mathcal{L}\left(\frac{\tan^2[\gamma(\tau)/2]\tan(\theta/2)}{\tan^2(\tau/2)}\right)$$

$$\times\frac{\partial}{\partial\gamma(\tau)}\int_{\gamma(\tau)}^{\pi}\frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\gamma(\tau)-\cos\theta_0}}\mathcal{L}\left(\cot\frac{\theta_0}{2}\right)v(\theta_0,\phi). \tag{1.5.33}$$

Interchanging the order of integration in (33), we obtain, after subsequent integration with respect to τ,

$$V(r,\theta,\phi) = \frac{a|r^2-a^2|}{2\pi^2}\int_0^{2\pi}d\phi_0\int_{\alpha}^{\pi}\left[\frac{R_0}{\chi}+\tan^{-1}\left(\frac{\chi}{R_0}\right)\right]\frac{v(\theta_0,\phi_0)}{R_0^3}\sin\theta_0\, d\theta_0, \tag{1.5.34}$$

where

$$\chi = y_1[t_1(\alpha)] = \frac{2\sqrt{ar}\sqrt{\cos t_1(\alpha)-\cos\theta}\sqrt{\cos\alpha-\cos\theta_0}}{\sin t_1(\alpha)}. \tag{1.5.35}$$

It can be proven that (35) can be obtained from (25) by a formal substitution of θ_0, θ, and α by $\pi-\theta_0$, $\pi-\theta$, and $\pi-\alpha$ respectively.

The derivation of (34) can be outlined as follows. By introducing a new variable $\tau=t_2(x)$, $x=\gamma(\tau)$, expression (33) can be reduced to

$$V(r,\theta,\phi)=-\frac{\sqrt{a}}{\pi\sqrt{r}}\int_{\alpha}^{\pi}\frac{t_2'(x)\,\mathrm{d}x}{\sqrt{\cos\theta-\cos t_2(x)}}\,\mathcal{L}\left(\frac{\tan^2[t_1(x)/2]}{\tan(\theta/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{\pi}\frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos x-\cos\theta_0}}\,\mathcal{L}\left(\cot\frac{\theta_0}{2}\right)v(\theta_0,\phi), \tag{1.5.36}$$

where

$$t_2'(x)=\frac{\partial t_2(x)}{\partial x}. \tag{1.5.37}$$

Integration by parts in (36) yields

$$V(r,\theta,\phi)=-\frac{\sqrt{a}}{\pi\sqrt{r}}\int_{\alpha}^{\pi}\left\{\frac{t_2'(\alpha)\,\mathcal{L}\left(\frac{\tan^2[t_1(\alpha)/2]}{\tan(\theta/2)\tan(\theta_0/2)}\right)}{\sqrt{\cos\theta-\cos t_2(\alpha)}\,\sqrt{\cos\alpha-\cos\theta_0}}\right.$$

$$\left.+\int_{\alpha}^{\theta_0}\frac{\mathrm{d}x}{\sqrt{\cos x-\cos\theta_0}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{t_2'(x)\,\mathcal{L}\left(\frac{\tan^2[t_1(x)/2]}{\tan(\theta/2)\tan(\theta_0/2)}\right)}{\sqrt{\cos\theta-\cos t_2(x)}}\right]\right\}v(\theta_0,\phi)\sin\theta_0\,\mathrm{d}\theta_0. \tag{1.5.38}$$

Introducing the function

$$F(x)=\frac{\sqrt{ar}\,|r^2-a^2|}{R_0^3}\left[\frac{R_0}{y_1[t_1(x)]}+\tan^{-1}\left(\frac{y_1[t_1(x)]}{R_0}\right)\right], \tag{1.5.39}$$

where y_1 and R_0 are defined by (9) and (10) respectively. We can prove the following identity:

$$\frac{t_2'(x)\,\lambda\left(\frac{\tan^2[t_1(x)/2]}{\tan(\theta/2)\tan(\theta_0/2)},\,\phi-\phi_0\right)}{\sqrt{\cos\theta-\cos t_2(x)}}=\frac{2(\cos x-\cos\theta_0)^{3/2}}{\sin x}\frac{\mathrm{d}F(x)}{\mathrm{d}x}. \tag{1.5.40}$$

In order to prove (40) one should use (5), (6), (11), and the following identities:

$$[\cos\theta-\cos t_2(x)][\cos t_1(x)-\cos\theta]=\left(\frac{a-r}{a+r}\right)^2\sin^2\theta,$$

$$[\cos x-\cos t_2(x)][\cos t_1(x)-\cos x]=\left(\frac{a-r}{a+r}\right)^2\sin^2 x,$$

$$\frac{\partial t_2(x)}{\partial x}=\frac{\sin\theta\,[\cos t_1(x)-\cos x]}{\sin x\,[\cos t_1(x)-\cos\theta]}\frac{\partial t_1(x)}{\partial x}=\frac{\sin x\,[\cos\theta-\cos t_2(x)]}{\sin\theta\,[\cos x-\cos t_2(x)]}\frac{\partial t_1(x)}{\partial x},$$

$$\sin t_1(x)\sin t_2(x)=\frac{4ar}{(a+r)^2}\sin x\sin\theta,$$

$$\frac{\sin t_1(x)}{\sin t_2(x)}=\frac{\sin\theta\,[\cos t_1(x)-\cos x]}{\sin x\,[\cos\theta-\cos t_2(x)]}=\frac{\sin x\,[\cos t_1(x)-\cos\theta]}{\sin\theta\,[\cos x-\cos t_2(x)]}$$

$$\cos t_1(x)\cos t_2(x)=\frac{4ar\cos\theta\cos x-(r-a)^2}{(r+a)^2},$$

$$\cos t_1(x)+\cos t_2(x)=\frac{4ar}{(r+a)^2}(\cos x+\cos\theta),$$

$$\frac{[\cos t_1(x)-\cos x]\,[\cos t_1(x)-\cos\theta]}{\sin^2 t_1(x)}=\frac{[\cos x-\cos t_2(x)]\,[\cos\theta-\cos t_2(x)]}{\sin^2 t_2(x)}=\frac{(a-r)^2}{4ar}.$$

Substitution of (40) in (38) yields

$$V(r,\theta,\phi)=\frac{\sqrt{a}}{2\pi^2\sqrt{r}}\int_0^{2\pi}\mathrm{d}\phi_0\int_\alpha^\pi\left\{\frac{2(\cos\alpha-\cos\theta_0)}{\sin\alpha}\frac{\mathrm{d}F(\alpha)}{\mathrm{d}\alpha}\right.$$

$$\left.+\int_\alpha^{\theta_0}\frac{\mathrm{d}x}{\sqrt{\cos x-\cos\theta_0}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2(\cos x-\cos\theta_0)^{3/2}}{\sin x}\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right]\right\}v(\theta_0,\phi_0)\sin\theta_0\,\mathrm{d}\theta_0. \qquad (1.5.41)$$

Integration by parts in (41) gives

$$V(r,\theta,\phi)=\frac{\sqrt{a}}{2\pi^2\sqrt{r}}\int_0^{2\pi}\mathrm{d}\phi_0\int_\alpha^\pi F(\alpha)\,v(\theta_0,\phi_0)\sin\theta_0\,\mathrm{d}\theta_0,$$

which is equivalent to (34). We note the analogy between (34) and (1.3.34). In the case $\alpha\to 0$ formula (34) transforms into the well-known Poisson's solution to the Dirichlet problem for a sphere.

Certain integral characteristics can be evaluated without solving any integral equation. For example, the total charge in Problem 1 can be found by integration of both sides of (17), with the result

$$Q_1=\frac{2}{\pi}a^2\int_0^{2\pi}\mathrm{d}\phi\int_0^\alpha \sigma(\theta,\phi)\cos^{-1}\left(\frac{\cos(\alpha/2)}{\cos(\theta/2)}\right)\sin\theta\,\mathrm{d}\theta. \tag{1.5.42}$$

The total charge in Problem 2 can be obtained from (29) as

$$Q_2=\frac{a}{2\pi^2}\int_0^{2\pi}\mathrm{d}\phi\int_\alpha^\pi\left[\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha-\cos\theta}}+\tan^{-1}\frac{\sqrt{\cos\alpha-\cos\theta}}{\sqrt{1-\cos\alpha}}\right]v(\theta,\phi)\sin\theta\,\mathrm{d}\theta. \tag{1.5.43}$$

Dirichlet problem with discontinuous boundary conditions. In many practical cases, the boundary conditions for Dirichlet problem are changing so rapidly that they can be modelled as discontinuous. The spherical harmonic expansion solution converges very badly in those cases, and it is usually divergent on the surface of the sphere, thus making the solution unfit for practical purposes. On the other hand, the closed form solution, given by Poisson, is very inconvenient for practical evaluation of of the integrals. The new method allows us to obtain an alternative solution, which is equivalent to the one obtained by Poisson, but is easy amenable for the exact evaluations of the integrals involved.

We consider a charged sphere with the following conditions at its surface $r=a$

$$V(a,\theta,\phi)=v(\theta,\phi),\ \text{for}\ \ 0\le\theta\le\alpha,\ \ 0\le\phi<2\pi;$$

$$V(a,\theta,\phi)=0,\ \text{for}\ \ \alpha\le\theta\le\pi,\ \ 0\le\phi<2\pi. \tag{1.5.44}$$

The problem, in a sense, is inverse to problem 1, therefore, substitution of (44) in (23) leads to the governing integral equation

$$2a\int_{\theta}^{\alpha}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{0}^{\tau}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)=v(\theta,\phi). \tag{1.5.45}$$

The exact solution of (45) can be obtained in a manner similar to that of (27). We apply the operator

$$\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)$$

to both sides of (45). The result is

$$-2\pi a\int_{0}^{\theta_1}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\theta_1}}\mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta_1/2)}\right)\sigma(\theta_0,\phi)=$$

$$\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \tag{1.5.46}$$

the next operator to apply is

$$\frac{\mathcal{L}[\cot(\theta_2/2)]}{\sin\theta_2}\frac{d}{d\theta_2}\int_{0}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan\frac{\theta_1}{2}\right),$$

thus giving the solution

$$\sigma(\theta_2,\phi)=\frac{\mathcal{L}[\cot(\theta_2/2)]}{2\pi^2 a\sin\theta_2}\frac{d}{d\theta_2}\int_{0}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \tag{1.5.47}$$

Expression (47) is valid in the interval $0 \le \theta_2 \le \alpha$. The charge density distribution at the rest of the sphere can be obtained by substitution of (47) in (17), with the result for $\alpha < \theta < \pi$

$$\sigma(\theta,\phi) = \frac{\mathcal{L}[\cot(\theta/2)]}{4\pi^2 a} \int_0^{\alpha} \frac{\sin\theta_1 \, d\theta_1}{(\cos\theta_1 - \cos\theta)^{3/2}} \mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times \frac{d}{d\theta_1} \int_{\theta_1}^{\alpha} \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\theta_1 - \cos\theta_0}} \mathcal{L}\left(\cot\frac{\theta_0}{2}\right) v(\theta_0,\phi). \tag{1.5.48}$$

The elementary analysis of (47) and (48) shows that both charge density distributions have non-integrable singularities of opposite sign at $\theta \to \alpha$, when $v(\alpha - 0, \phi) \neq 0$, otherwise expression (47) has no singularities, and formula (48) can give an integrable singularity. The total charge can be obtained by integration (47) and (48), with the result

$$Q = \frac{a}{4\pi} \int_0^{2\pi} d\phi \int_0^{\alpha} v(\theta,\phi) \sin\theta \, d\theta. \tag{1.5.49}$$

The potential in space due to a charged sphere can be obtained by substitution of (47) in (23) which yields, after simplification,

$$V(r,\theta,\phi) = -\frac{\sqrt{a}}{\pi\sqrt{r}} \int_{t_2(0)}^{t_2(\alpha)} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan^2[\gamma(\tau)/2]}{\tan^2(\tau/2)}\right)$$

$$\times \frac{\partial}{\partial\gamma(\tau)} \int_{\gamma(\tau)}^{\alpha} \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}} \mathcal{L}\left(\cot\frac{\theta_0}{2}\right) v(\theta_0,\phi). \tag{1.5.50}$$

Interchange of the order of integration in (50) and subsequent integration with respect to τ result in the well-known Poisson formula, namely,

$$V(r,\theta,\phi) = -\frac{a|r^2 - a^2|}{4\pi} \int_0^{2\pi} d\phi_0 \int_0^{\alpha} \frac{v(\theta_0,\phi_0)}{R_0^3} \sin\theta_0 \, d\theta_0. \tag{1.5.51}$$

Expression (50) is equivalent to (51) and has definite advantages when an exact evaluation of the integrals is possible.

Influence of a point charge. We consider the interaction between a point charge q located at the point with spherical coordinates (r_0, θ_0, ϕ_0) and a grounded spherical cap $\alpha \le \theta \le \pi$ of radius a. The potential in space can be represented as a sum

$$V = V_q + V_c, \tag{1.5.52}$$

where V_q is the potential due to the point charge q, and V_c is the potential of the charge induced on the spherical cap. At the surface of the cap holds the condition $V=0$, which implies that

$$V_c = -V_q = -q/R, \tag{1.5.53}$$

with

$$R^2 = r_0^2 + a^2 - 2ar_0\,[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)].$$

Now we have a mixed problem with the following conditions at the surface of the sphere $r=a$:

$$\sigma(\theta,\phi) = 0, \quad \text{for} \quad 0\le\theta<\alpha,\ \ 0\le\phi<2\pi;$$

$$V(a,\theta,\phi) = -q/R, \quad \text{for} \quad \alpha\le\theta\le\pi,\ \ 0\le\phi<2\pi. \tag{1.5.54}$$

Conditions (54) correspond to Problem 2, which has already been solved. According to (4), we can write the following integral representation

$$\frac{1}{R} = \frac{1}{\pi\sqrt{ar_0}} \int_{t_{20}(\theta)}^{\pi} \frac{\lambda\left(\dfrac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\theta_0-\cos\tau}\,\sqrt{\cos\theta-\cos\gamma_0(\tau)}}, \tag{1.5.55}$$

where, according to (5) and (6)

$$t_{20}(\theta) \equiv t_2(\theta,\theta_0,a,r_0), \quad \cos\gamma_0(\tau) = \cos\tau - \frac{(r_0-a)^2}{4ar_0}\,\frac{\sin^2\tau}{\cos\tau-\cos\theta_0}, \tag{1.5.56}$$

The induced charge density distribution is given by (29) which after substitution of (55) simplifies for $\alpha<\theta<\pi$ to

$$\sigma(\theta,\phi) = -q\,\frac{\mathcal{L}[\cot(\theta/2)]}{2\pi^2 a\sin\theta}\frac{d}{d\theta}\int_\alpha^\theta \frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta}}$$

$$\times\lambda\left(\frac{\tan^2(\theta_1/2)\tan(\theta_0/2)}{\tan^2[t_{20}(\theta_1)/2]},\phi-\phi_0\right)\frac{\partial t_{20}(\theta_1)/\partial\theta_1}{\sqrt{ar_0}\sqrt{\cos\theta_0-\cos t_{20}(\theta_1)}}. \tag{1.5.57}$$

The integral in (57) can be evaluated exactly in the same manner as before, with the result

$$\sigma(\theta,\phi) = -\frac{q|r^2-a^2|}{2\pi^2 aR^3}\left[\frac{R}{\chi_0}+\tan^{-1}\frac{\chi_0}{R}\right], \tag{1.5.58}$$

where

$$\chi_0 = \frac{2\sqrt{ar_0}\sqrt{\cos t_{10}(\alpha)-\cos\theta_0}\sqrt{\cos\alpha-\cos\theta}}{\sin t_{10}(\alpha)}, \tag{1.5.59}$$

$$t_{10}(x)\equiv t_1(x,\theta_0,a,r_0),\quad t_{20}(x)\equiv t_2(x,\theta_0,a,r_0),$$

Note certain similarity between (58)–(59) and (34)–(35). In the particular case $r_0\to a$ and $\theta_0<\alpha$, formula (58) simplifies as

$$\sigma(\theta,\phi) = -q\,\frac{\sqrt{\cos\theta_0-\cos\alpha}}{2\pi^2\sqrt{\cos\alpha-\cos\theta}}\frac{1}{R^2}, \tag{1.5.60}$$

which is in agreement with (18). Expression (58) is convenient for a direct evaluation of the induced charge density distribution but, if some further mathematical transformations are needed, then the equivalent formula (57) has definite advantages. For example, to evaluate the total charge Q using (58) would be quite difficult, while (57) immediately gives

$$Q = -\frac{q\sqrt{a}}{\pi\sqrt{r_0}}\int_\alpha^\pi \frac{\sin\theta_1}{\sqrt{1+\cos\theta_1}\sqrt{\cos\theta_0-\cos t_{20}(\theta_1)}}\frac{\partial t_{20}(\theta_1)}{\partial\theta_1}\,d\theta_1. \tag{1.5.61}$$

Introducing a new variable $\tau = t_{20}(\theta_1)$, $\theta_1 = \gamma_0(\tau)$ in (61) the following simplification occurs:

$$Q = -\frac{q\sqrt{a}}{\pi\sqrt{r_0}}\int_{t_{20}(\alpha)}^{\pi}\frac{\sqrt{1-\cos\gamma_0(\tau)}}{\sqrt{\cos\theta_0-\cos\tau}}\,d\tau, \tag{1.5.62}$$

the evaluation of which is elementary, and the final result is

$$Q = -\frac{q}{\pi r_0}[(a+r_0)\sin^{-1}A_{10} - |a-r_0|\sin^{-1}A_{20}] \tag{1.5.63}$$

where

$$A_{10} = \frac{(r_0+a)\cos(\alpha/2)}{\sqrt{m_{20}^2(\alpha)+4ar_0\cos^2(\theta/2)\cos^2(\alpha/2)}},$$

$$A_{20} = \frac{|r_0-a|\cos(\alpha/2)}{\sqrt{m_{20}^2(\alpha)-4ar_0\sin^2(\theta/2)\cos^2(\alpha/2)}}, \quad m_{20}(\alpha) = m_2(\alpha,\theta,a,r_0). \tag{1.5.64}$$

When the point charge is located at the axis ($\theta_0 = \pi$), formula (63) simplifies as

$$Q = \frac{q}{\pi r_0}\left[|r_0-a|\left(\frac{\pi-\alpha}{2}\right) - (r_0+a)\tan^{-1}\left(\frac{r_0+a}{|r_0-a|}\cot\frac{\alpha}{2}\right)\right]. \tag{1.5.65}$$

In the case of a complete sphere we have $\alpha = 0$, and formula (65) simplifies further

$$Q = \frac{q}{2r_0}[|r_0-a| - (r_0+a)].$$

Similar formulae can be obtained for $\theta_0 = 0$.

The potential V_c due to the induced charge can be obtained by substitution of (55) in (33) which gives, after the first integration

$$V_c(r,\theta,\phi) = -\frac{q}{\pi\sqrt{rr_0}}\int_{\alpha}^{\pi}\frac{\lambda\left(\dfrac{\tan(t_1/2)\tan(t_{10}/2)}{\tan(t_2/2)\tan(t_{20}/2)}, \phi-\phi_0\right)t_2'\,t_{20}'\,dx}{\sqrt{\cos\theta-\cos t_2}\sqrt{\cos\theta_0-\cos t_{20}}}. \tag{1.5.66}$$

Here the abbreviations t_1, t_2, t_{10}, and t_{20} are understood as $t_1(x)$, $t_2(x)$, $t_{10}(x)$, $t_{20}(x)$ respectively, the prime signs indicate the partial derivatives with respect to x. The integral in (66) can be evaluated exactly, and the final result is

$$V(r,\theta,\phi) = V_q + V_c = q\left\{ \frac{1}{2R_1}\left[1 + \frac{2}{\pi}\tan^{-1}\frac{\eta_1(\alpha)}{R_1}\right] - \frac{1}{2R_2}\left[1 - \frac{2}{\pi}\tan^{-1}\frac{\eta_2(\alpha)}{R_2}\right]\right\}, \tag{1.5.67}$$

which is in agreement with a similar result of Hobson (1900) in toroidal coordinates. The notations are

$$R_1^2 = r^2 + r_0^2 - 2\,rr_0[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)],$$

$$R_2^2 = \frac{r^2 r_0^2}{a^2} + a^2 - 2\,rr_0[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)],$$

$$\eta_{1,2} = \frac{(r+a)(r_0+a)}{2a}\,S(x) \pm \frac{(r-a)(r_0-a)}{2aS(x)},$$

$$S(x) = \sqrt{\cos t_1 - \cos x}\ \ \sqrt{\cos t_{10} - \cos x}/\sin x. \tag{1.5.68}$$

The following identities were used to perform integration in (66)

$$\lambda\left(\frac{\tan(t_1/2)\tan(t_{10}/2)}{\tan(t_2/2)\tan(t_{20}/2)}, \phi - \phi_0\right) = \frac{(a+r)^2\,(a+r_0)^2}{16a^2\sin^2 x}\,[(1 - \cos t_1\cos t_2)(\cos t_{10} - \cos t_{20})$$

$$+ (1 - \cos t_{10}\cos t_{20})(\cos t_1 - \cos t_2)]\left[\frac{1}{R_1^2 + \eta_1^2(x)} + \frac{1}{R_2^2 + \eta_2^2(x)}\right], \tag{1.5.69}$$

$$R_1^2 + \eta_1^2(x) = R_2^2 + \eta_2^2(x), \tag{1.5.70}$$

$$\frac{\partial}{\partial x}S(x) = \frac{S(x)}{2\sin x}\left[\frac{1 - \cos t_1\cos t_2}{\cos t_1 - \cos t_2} + \frac{1 - \cos t_{10}\cos t_{20}}{\cos t_{10} - \cos t_{20}}\right], \tag{1.5.71}$$

$$\frac{\partial t_2}{\partial x} = \frac{2\sqrt{ar}}{a+r}\,\frac{\sqrt{\cos t_1 - \cos x}\sqrt{\cos\theta - \cos t_2}}{\cos t_1 - \cos t_2}. \tag{1.5.72}$$

An expression similar to (72) can be written for the derivative of t_{20}. Substitution of (69)–(72) in (66) makes the procedure of integration very simple.

1.6. Mixed problems in toroidal coordinates

Further extension of previously obtained results to the case of toroidal coordinates is presented here. It is based on a new integral representation for the reciprocal of the distance between two points. Its substitution in the governing integral equation reduces the problem to sequence of two consecutive Abel type operators combined with the $\mathcal{L}$-operator. Each can be inverted exactly and in closed form, thus giving the solution. Some integrals of fundamental value, involving distances between several points, are established. The complete set of systems of coordinates, where the new method can be applied, is not known at this time and can constitute a subject for a separate investigation.

Mathematical preliminaries. The following relationships exist between the cartesian (x, y, z) and toroidal (v, u, ϕ) coordinates

$$x = \frac{c\sinh v\cos\phi}{\cosh v - \cos u}, \quad y = \frac{c\sinh v\sin\phi}{\cosh v - \cos u}, \quad z = \frac{c\sin u}{\cosh v - \cos u}. \tag{1.6.1}$$

Here c is a dimensional parameter. The surfaces $u=$constant are spherical caps

$$x^2 + y^2 + (z - c\cot u)^2 = \left(\frac{c}{\sin u}\right)^2, \tag{1.6.2}$$

with the common line of intersection along the circle $\rho = c$, $z = 0$. The surfaces $v=$constant are tori

$$(\sqrt{x^2+y^2} - c\coth v)^2 + z^2 = \left(\frac{c}{\sinh v}\right)^2. \tag{1.6.3}$$

The properties of toroidal coordinates allow us to use this system of coordinates for solving mixed boundary value problems for various geometries including the case of several spherical caps.

Consider two points M and N in a three-dimensional space. Let their cylindrical coordinates be respectively (ρ, ϕ, z) and (r, ψ, z_0). From the results of section 1.2, the following integral is valid

$$\int \frac{\lambda\left(\dfrac{y^2}{l_1(r)l_2(r)}, \phi - \psi\right)\mathrm{d}y}{\sqrt{l_1^2(r) - y^2}\sqrt{l_2^2(r) - y^2}} = -\frac{1}{R_0}\tan^{-1}\left[\frac{\sqrt{l_1^2(r) - y^2}\sqrt{l_2^2(r) - y^2}}{y R_0}\right]. \tag{1.6.4}$$

Here, as before,

$$\lambda(k,\psi)=\frac{1-k^2}{1+k^2-2k\cos\psi},\quad \text{for } k<1, \tag{1.6.5}$$

$$l_1(r)=\frac{1}{2}[\sqrt{(\rho+r)^2+(z-z_0)^2}-\sqrt{(\rho-r)^2+(z-z_0)^2}],$$

$$l_2(r)=\frac{1}{2}[\sqrt{(\rho+r)^2+(z-z_0)^2}+\sqrt{(\rho-r)^2+(z-z_0)^2}], \tag{1.6.6}$$

$$R_0\equiv R(M,N)=[\rho^2+r^2-2\rho r\cos(\phi-\psi)+(z-z_0)^2]. \tag{1.6.7}$$

The integral representation for reciprocal of the distance between two points can be obtained from (4) by applying the limits of integration from 0 to $l_1(r)$. This representation is fundamental for the new method in cylindrical coordinates. One can verify that the distance between two points $R(M,N)$ can be presented as

$$R_0=\sqrt{l_1^2(r)+l_2^2(r)-2l_1(r)l_2(r)\cos(\phi-\psi)}. \tag{1.6.8}$$

In order to be able to apply (4) in toroidal coordinates, we need to present the distance between two points in a similar form, namely, as a sum of two squares minus double product of those quantities and cosine of the difference between the appropriate angles. Let the toroidal coordinates of M and N be (v,u,ϕ) and (x,β,ψ) respectively. The distance between two points in toroidal coordinates is

$$R_0\equiv R(M,N)=\frac{\sqrt{2}c\sqrt{\cosh v\cosh x-\sinh v\sinh x\cos(\phi-\psi)-\cos(u-\beta)}}{\sqrt{\cosh v-\cos u}\,\sqrt{\cosh x-\cos\beta}}. \tag{1.6.9}$$

Clearly, (9) does not look like (8), but we can transform it into

$$R_0=\frac{2c\cosh(v/2)\cosh(x/2)}{\sqrt{\cosh v-\cos u}\,\sqrt{\cosh x-\cos\beta}}\left[\tanh^2\left(\frac{v}{2}\right)+\tanh^2\left(\frac{x}{2}\right)\right.$$

$$\left.-2\tanh\left(\frac{v}{2}\right)\tanh\left(\frac{x}{2}\right)\cos(\phi-\psi)+\frac{\sin^2[(u-\beta)/2]}{\cosh^2(v/2)\cosh^2(x/2)}\right]^{1/2}. \tag{1.6.10}$$

Expression (10) gives us a hint that we can introduce some quantities t_1 and t_2 in such a way that the distance between two points will be proportional to the expression

$$\left[\tanh^2\left(\frac{t_1}{2}\right)+\tanh^2\left(\frac{t_2}{2}\right)-2\tanh\left(\frac{t_1}{2}\right)\tanh\left(\frac{t_2}{2}\right)\cos(\phi-\psi)\right]^{1/2},$$

so that these quantities could play in toroidal coordinates the same role as the parameters l_1 and l_2 play in cylindrical coordinates. Indeed, this can be achieved by defining

$$t_1 = 2\tanh^{-1}\left[\frac{l_1}{l_2}\tanh\left(\frac{v}{2}\right)\tanh\left(\frac{x}{2}\right)\right]^{1/2},$$

$$t_2 = 2\tanh^{-1}\left[\frac{l_2}{l_1}\tanh\left(\frac{v}{2}\right)\tanh\left(\frac{x}{2}\right)\right]^{1/2}. \tag{1.6.11}$$

We can now introduce a new variable τ according to the expression

$$y = \left[l_1 l_2 \coth\left(\frac{v}{2}\right)\coth\left(\frac{x}{2}\right)\right]^{1/2}\tanh\left(\frac{\tau}{2}\right). \tag{1.6.12}$$

Substitution of (12) in (4) yields

$$\int \frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}, \phi-\psi\right)d\tau}{\sqrt{\cosh v - \cosh\tau}\sqrt{\cosh x - \cosh\gamma(\tau)}} = -\frac{2c}{R_0\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}$$

$$\times\tan^{-1}\left[\frac{2c\sqrt{\cosh v - \cosh\tau}\sqrt{\cosh x - \cosh\gamma(\tau)}}{R_0\sinh\tau\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}\right]. \tag{1.6.13}$$

Here

$$\cosh\gamma \equiv \cosh\gamma(\tau) \equiv \cosh\gamma(\tau, \beta, v, u) = \cosh\tau + \sin^2\left(\frac{u-\beta}{2}\right)\frac{\sinh^2\tau}{\cosh v - \cosh\tau}, \tag{1.6.14}$$

We intentionally use in this section the same notation t_1, t_2, and γ in order to demonstrate certain analogy between the toroidal and spherical coordinates. We hope the reader will not be confused. Introduce the following notation

$$t_1 \equiv t_1(x, \beta, v, u)$$

$$= 2\tanh^{-1}\left\{\frac{\sqrt{\cosh(x+v) - \cos(u-\beta)} - \sqrt{\cosh(x-v) - \cos(u-\beta)}}{2\sqrt{2}\cosh(x/2)\cosh(v/2)}\right\}, \tag{1.6.15}$$

$$t_2 \equiv t_2(x, \beta, v, u)$$

$$= 2\tanh^{-1}\left\{\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)}+\sqrt{\cosh(x-v)-\cos(u-\beta)}}{2\sqrt{2}\cosh(x/2)\cosh(v/2)}\right\},$$

(1.6.16)

For the brevity sake, we use the following conventions: the parameters of γ, t_1 and t_2, given respectively in (14), (15), and (16), are considered as the default parameters. This would allow us to write, for example, $\gamma(y,\delta)$ instead of $\gamma(y,\delta,v,u)$. The rule is rather simple: the parameters which are not given explicitly assumed to be the default ones.

One can verify that (15) and (16) are in agreement with (11). Notice that both t_1 and t_2 are inverse to γ. This means that $\gamma(t_1)=x$ and $\gamma(t_2)=x$. The following property is valid $t_1\le\min\{v,x\}$, $t_2\ge\max\{v,x\}$, the equality sign holds for $u=\beta$. By using previous results we can obtain the following integral representation for the reciprocal of the distance between two points:

$$\frac{1}{R_0}=\frac{\sqrt{\cosh v-\cos u}\sqrt{\cosh x-\cos\beta}}{\pi c}\int_0^{t_1}\frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)},\phi-\psi\right)d\tau}{\sqrt{\cosh v-\cosh\tau}\sqrt{\cosh x-\cosh\gamma}}.$$

(1.6.17)

We can derive several variations of (17). For example, introducing a new variable $\tau=t_1(y)$, expression (17) will take the form

$$\frac{1}{R_0}=\frac{\sqrt{\cosh v-\cos u}\sqrt{\cosh x-\cos\beta}}{\pi c}\int_0^{x}\frac{\lambda\left(\dfrac{\tanh^2(t_1(y)/2)}{\tanh(v/2)\tanh(x/2)},\phi-\psi\right)t_1'(y)\,dy}{\sqrt{\cosh v-\cosh t_1(y)}\sqrt{\cosh x-\cosh y}}.$$

(1.6.18)

Here the symbol (´) stands for the partial derivative with respect to the parameter in brackets. By using (A12), one can rewrite (18) as

$$\frac{1}{R_0}=\frac{\sqrt{\cosh v-\cos u}\sqrt{\cosh x-\cos\beta}}{\pi c\,|\cos[(u-\beta)/2]|}$$

$$\times\int_0^{x}\frac{\lambda\left(\dfrac{\tanh^2(t_1(y)/2)}{\tanh(v/2)\tanh(x/2)},\phi-\psi\right)\sqrt{\cosh t_2(y)-\cosh y}\,dy}{[\cosh t_2(y)-\cosh t_1(y)]\sqrt{\cosh x-\cosh y}}. \quad (1.6.19)$$

We can also compute a more general indefinite integral, namely,

$$I_1 = \frac{1}{\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]}$$

$$\times \int \frac{\sqrt{\cosh t_2 - \cosh x}\sqrt{\cosh t_{20} - \cosh x}}{(\cosh t_2 - \cosh t_1)(\cosh t_{20} - \cosh t_{10})} \lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) \mathrm{d}x \,. \tag{1.6.20}$$

Here $t_{10} = t_1(x, \beta, v_0, u_0)$ and $t_{20} = t_2(x, \beta, v_0, u_0)$ respectively. Introduce new variables

$$\eta_{1,2} = \cos\left(\frac{u-\beta}{2}\right)\cos\left(\frac{u_0-\beta}{2}\right)S(x) \pm \frac{1}{S(x)}\sin\left(\frac{u-\beta}{2}\right)\sin\left(\frac{u_0-\beta}{2}\right), \tag{1.6.21}$$

where

$$S(x) = \frac{\sqrt{\cosh t_2 - \cosh x}\sqrt{\cosh t_{20} - \cosh x}}{\sinh x} \tag{1.6.22}$$

The following identities may be established by using formulae from Appendix:

$$\frac{\mathrm{d}S(x)}{\mathrm{d}x} = -\frac{S(x)}{2\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.6.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{S(x)}\right) = \frac{1}{2\,S(x)\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.6.24}$$

$$\lambda\left(\frac{\tanh(t_1/2\tanh(t_{10}/2))}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) = \frac{\cos^2[(u-\beta)/2]\cos^2[(u_0-\beta)/2]}{2\sinh^2 x}\left[\frac{\cosh t_1\cosh t_2 - 1}{\cosh t_2 - \cosh t_1}\right.$$

$$\left. + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right](\cosh t_2 - \cosh t_1)$$

$$\times (\cosh t_{20} - \cosh t_{10})\left[\frac{1}{\cosh w - \cos(u - u_0) + 2\eta_1^2}\right.$$

$$\left. + \frac{1}{\cosh w - \cos(u + u_0 - 2\beta) + 2\eta_2^2}\right]. \tag{1.6.25}$$

Here

$$\cosh w = \cosh v \cosh v_0 - \sinh v \sinh v_0 \cos(\phi - \phi_0). \tag{1.6.26}$$

The transformations leading to (23)–(25) are very non-trivial. One has to use the appropriate formulae from Appendix in an ingenious way in order to repeat the results. Taking into consideration that

$$\frac{d\eta_1(x)}{dx} = \frac{\eta_2(x)}{S(x)}\frac{dS(x)}{dx}, \qquad \frac{d\eta_2(x)}{dx} = \frac{\eta_1(x)}{S(x)}\frac{dS(x)}{dx},$$

$$\cosh w - \cos(u-u_0) + 2\eta_1^2 = \cosh w - \cos(u+u_0-2\beta) + 2\eta_2^2, \tag{1.6.27}$$

the substitution of (23)–(25) and (27) in (20) leads to

$$I_1 = -\int \left[\frac{d\eta_1}{\cosh w - \cos(u-u_0) + 2\eta_1^2} + \frac{d\eta_2}{\cosh w - \cos(u+u_0-2\beta) + 2\eta_2^2}\right]. \tag{1.6.28}$$

The last integral can be computed in an elementary way, and the final result is

$$I_1 = -\frac{1}{\sqrt{2[\cosh w - \cos(u-u_0)]}}\tan^{-1}\left[\frac{\sqrt{2}\eta_1(x)}{\sqrt{\cosh w - \cos(u-u_0)}}\right]$$
$$-\frac{1}{\sqrt{2[\cosh w - \cos(u+u_0-2\beta)]}}\tan^{-1}\left[\frac{\sqrt{2}\eta_2(x)}{\sqrt{\cosh w + \cos(u+u_0-2\beta)}}\right]. \tag{1.6.29}$$

Here the reader may ask us two questions. First, why have we decided that the integral (20) is computable, and second, how did we come up with expressions (21) and the properties (23)–(25)? The integral (20) was encountered in solving the problem of influence of a point charge on a spherical bowl which, as we know, has an elementary solution. This meant that the integral (20) has to be computable. The hints on how to compute it can be taken from similar integral in section 1.5. One has just to replace the appropriate trigonometric functions by the hyperbolic ones.

Yet another integral can be computed in a similar manner, namely,

$$I_2 = \frac{1}{\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]}$$
$$\times\int \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh x - \cosh t_{10}}}{(\cosh t_2 - \cosh t_1)(\cosh t_{20} - \cosh t_{10})}\lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi-\phi_0\right)dx. \tag{1.6.30}$$

The same integral (30) can be rewritten as

$$I_2 = \int \lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi-\phi_0\right)\frac{t_2'(x)t_{20}'(x)\,\mathrm{d}x}{\sqrt{\cosh t_2-\cosh v}\,\sqrt{\cosh t_{20}-\cosh v_0}}. \tag{1.6.31}$$

Introduction of new variables

$$\Theta_{1,2} = \cos\left(\frac{u-\beta}{2}\right)\cos\left(\frac{u_0-\beta}{2}\right)T(x) \pm \frac{1}{T(x)}\sin\left(\frac{u-\beta}{2}\right)\sin\left(\frac{u_0-\beta}{2}\right), \tag{1.6.32}$$

with

$$T(x) = \frac{\sqrt{\cosh x-\cosh t_1}\sqrt{\cosh x-\cosh t_{10}}}{\sinh x},$$

and use of the identities (25) and

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = \frac{T(x)}{2\sinh x}\left[\frac{\cosh t_1\cosh t_2-1}{\cosh t_2-\cosh t_1}+\frac{\cosh t_{10}\cosh t_{20}-1}{\cosh t_{20}-\cosh t_{10}}\right], \tag{1.6.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{T(x)}\right) = -\frac{1}{2\,T(x)\sinh x}\left[\frac{\cosh t_1\cosh t_2-1}{\cosh t_2-\cosh t_1}+\frac{\cosh t_{10}\cosh t_{20}-1}{\cosh t_{20}-\cosh t_{10}}\right], \tag{1.6.34}$$

allow us to compute the integral

$$I_2 = \frac{1}{\sqrt{2[\cosh w-\cos(u-u_0)]}}\tan^{-1}\left[\frac{\sqrt{2}\Theta_1(x)}{\sqrt{\cosh w-\cos(u-u_0)}}\right]$$

$$+\frac{1}{\sqrt{2[\cosh w-\cos(u+u_0)-2\beta]}}\tan^{-1}\left[\frac{\sqrt{2}\Theta_2(x)}{\sqrt{\cosh w+\cos(u+u_0-2\beta)}}\right]. \tag{1.6.35}$$

One may deduce from (A2) that

$$T(x) = |\tan\left(\frac{u-\beta}{2}\right)\tan\left(\frac{u_0-\beta}{2}\right)\frac{1}{S(x)}.$$

This property gives us various relationships between η and Θ depending on the signs of the trigonometric functions. For example, when $\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]>0$ and $\sin[(u-\beta)/2]\sin[(u_0-\beta)/2]>0$, we have $\eta_1=\Theta_1$ and $\eta_2=-\Theta_2$. The derived integrals will be used in solving various mixed boundary value problems.

Problem description. Consider two spherical caps defined in the toroidal coordinates (v,u,ϕ) as follows:

$$0\le v\le b_0, \qquad u=u_0, \qquad 0\le\phi\le 2\pi;$$

$$0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi. \tag{1.6.36}$$

In the limiting case $b \to \infty$ and $b_0 \to \infty$ the spherical caps intersect along a circle of radius c which is the basic circle of the system of coordinates. Consider an electrostatic problem when an arbitrary charge distribution σ is prescribed on the first spherical cap ($u = u_0$), and an arbitrary potential distribution V is given on the surface of the second cap. It is then necessary to find the electrostatic field in the whole space. It is convenient to split the problem in two: the first problem assumes that $\sigma = 0$ and $V \neq 0$, while in the second problem we take $V = 0$ and $\sigma \neq 0$. The linear superposition of the two solutions would give us the general solution to the problem.

Problem 1. Since the first cap is not charged, we have to solve the Dirichlet problem for a spherical cap with the following condition on its surface:

$$V = V(v, \phi), \text{ for } 0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi. \tag{1.6.37}$$

The as yet unknown potential in space can be represented through a simple layer distribution

$$V(v, u, \phi) = c^2 \int_0^{2\pi} \int_0^b \frac{\sigma(x, \psi) \sinh x \, dx \, d\psi}{(\cosh x - \cos\beta)^2 R_0}. \tag{1.6.38}$$

Here σ is the charge distribution and R_0 is defined by (9). Substitution of (17) in (38) yields, after interchanging the order of integration

$$V(v, u, \phi) = 2c\sqrt{\cosh v - \cos u} \int_0^{t_1(b)} \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\times \int_\gamma^b \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x, \phi) \sinh x \, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\gamma}}. \tag{1.6.39}$$

The following scheme of interchanging the order of integration was used in (39):

$$\int_0^b dx \int_0^{t_1} d\tau = \int_0^{t_1(b)} d\tau \int_\gamma^b dx. \tag{1.6.40}$$

Substituting the boundary condition (37) in (39) results in the governing integral equation

$$V(v,\phi) = 2c\sqrt{\cosh v - \cos\beta}\int_0^v \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\times \int_\tau^b \frac{\mathcal{L}\left(\frac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}. \tag{1.6.41}$$

The integral equation (41) represents a sequence of two Abel type operators and the $\mathcal{L}$-operator. Each can be inverted in a manner similar to the one employed in previous sections. Let us apply the following operator

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\sinh v\, dv}{2c\sqrt{\cosh v - \cos\beta}\sqrt{\cosh y - \cosh v}}$$

to both sides of (41) in the following manner:

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\, dv}{2c\sqrt{\cosh v - \cos\beta}\sqrt{\cosh y - \cosh v}} = \mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}$$

$$\times \int_0^y \frac{\sinh v\, dv}{\sqrt{\cosh y - \cosh v}}\int_0^v \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}\int_\tau^b \frac{\mathcal{L}\left(\frac{\tanh^2(\tau/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}. \tag{1.6.42}$$

By using the general property

$$\int_0^y \frac{\sinh x\, dx}{\sqrt{\cosh y - \cosh x}}\int_0^{t_1} \frac{f(\tau)\, d\tau}{\sqrt{\cosh x - \cosh\gamma}} = \pi\int_0^{t_1(y)} f(\tau)\; d\tau, \tag{1.6.43}$$

expression (42) can be simplified, namely,

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}$$

$$=\pi\int_y^b \frac{\mathcal{L}\left(\frac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh y}}. \tag{1.6.44}$$

The next operator to apply to both sides of (44) is

$$\mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y-\cosh s}}\,\mathcal{L}\left(\coth\frac{y}{2}\right). \tag{1.6.45}$$

The final result is

$$\sigma(s,\phi)=-\frac{(\cosh s-\cos\beta)^{3/2}}{2\pi^2 c\sinh s}\,\mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y-\cosh s}}\,\mathcal{L}\left(\coth^2\frac{y}{2}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{\sqrt{\cosh v-\cos\beta}\,\sqrt{\cosh y-\cosh v}}. \tag{1.6.46}$$

Here the following property was used

$$\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y-\cosh s}}\int_y^b \frac{f(x)\,\mathrm{d}x}{\sqrt{\cosh x-\cosh y}}=\pi\int_s^b f(x)\,\mathrm{d}x. \tag{1.6.47}$$

Formula (46) gives the expression for the charge density in terms of the prescribed potential V.

We can now substitute (46) in (39) in order to obtain the potential in space through its value on the spherical cap. By using the property

$$\int_\gamma^b \frac{\mathrm{d}x}{\sqrt{\cosh x-\cosh\gamma}}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{f(v)\sinh v\,\mathrm{d}v}{\sqrt{\cosh v-\cosh x}}=-\pi f(\gamma), \tag{1.6.48}$$

the following result can be obtained

$$V(v,u,\phi)=\frac{1}{\pi}\sqrt{\cosh v-\cos u}\int_0^{t_1(b)}\frac{\mathrm{d}\tau}{\sqrt{\cosh v-\cosh\tau}}\,\mathcal{L}\left(\frac{\tanh^2(\tau/2)}{\tanh^2(\gamma/2)\tanh(v/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}\gamma}\int_0^{\gamma}\frac{\sinh y\,\mathcal{L}[\tanh(y/2)]\,V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y-\cos\beta}\sqrt{\cosh\gamma-\cosh y}}. \tag{1.6.49}$$

Introduction in (49) of a new variable $x=\gamma$, (which is equivalent to $\tau=t_1$), allows us to rewrite (49) as

$$V(v,u,\phi)=\frac{1}{\pi}\sqrt{\cosh v-\cos u}\int_0^{b}\frac{\mathrm{d}t_1}{\sqrt{\cosh v-\cosh t_1}}\,\mathcal{L}\left(\frac{\tanh(v/2)}{\tanh^2(t_2/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x}\frac{\sinh y\,\mathcal{L}[\tanh(y/2)]\,V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y-\cos\beta}\sqrt{\cosh x-\cosh y}}. \tag{1.6.50}$$

We interchange the order of integration in (50) according to the scheme

$$\int_0^b F(x)\,\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{\sinh y\,f(y)\,\mathrm{d}y}{\sqrt{\cosh x-\cosh y}}=-\int_0^b f(y)\,\mathrm{d}y\,\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b\frac{\sinh x\,F(x)\,\mathrm{d}x}{\sqrt{\cosh x-\cosh y}}. \tag{1.6.51}$$

The result is

$$V(v,u,\phi)=-\frac{1}{\pi}\sqrt{\cosh v-\cos u}\int_0^b\mathcal{L}\left(\tanh\frac{y}{2}\right)\left[\frac{\mathrm{d}}{\mathrm{d}y}\right.$$

$$\left.\times\int_y^b\frac{t_1'(x)\sinh x\,\mathcal{L}\left(\frac{\tanh(v/2)}{\tanh^2(t_2/2)}\right)\mathrm{d}x}{\sqrt{\cosh x-\cosh y}\sqrt{\cosh v-\cosh t_1}}\right]\frac{V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y-\cos\beta}}. \tag{1.6.52}$$

The interior integral in (52) can be computed in closed form. Indeed, consider the expression

$$I_3 = \mathcal{L}\left(\tanh\frac{y}{2}\right)\left[\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{t_1'(x)\sinh x\,\lambda\left(\frac{\tanh(v/2)}{\tanh^2(t_2/2)},\phi-\psi\right)\mathrm{d}x}{\sqrt{\cosh x-\cosh y}\,\sqrt{\cosh v-\cosh t_1}}\right] \tag{1.6.53}$$

The differentiation can be performed according to the rule

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{f(x)\sinh x\,\mathrm{d}x}{\sqrt{\cosh x-\cosh y}} = -\frac{f(b)\sinh y}{\sqrt{\cosh b-\cosh y}} + \sinh y\int_y^b \frac{\mathrm{d}f(x)}{\sqrt{\cosh x-\cosh y}}, \tag{1.6.54}$$

with the result

$$I_3 = -\frac{t_1'(b)\,\lambda\left(\frac{\tanh(y/2)\tanh(v/2)}{\tanh^2[t_2(b)/2]},\phi-\psi\right)\sinh y}{\sqrt{\cosh v-\cosh t_1(b)}}$$

$$+\sinh y\int_y^b \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{t_1'(x)\,\lambda\left(\frac{\tanh(y/2)\tanh(v/2)}{\tanh^2(t_2/2)},\phi-\psi\right)}{\sqrt{\cosh v-\cosh t_1}}\right\}\frac{\mathrm{d}x}{\sqrt{\cosh x-\cosh y}} \tag{1.6.55}$$

Introduce the following notation:

$$F(x) = \frac{4c^3\,|\sin(u-\beta)|}{(\cosh v-\cos u)^{3/2}(\cosh y-\cos\beta)^{3/2}}\frac{1}{R_y^3}\left[\frac{R_y}{\chi(x)}+\tan^{-1}\left(\frac{\chi(x)}{R_y}\right)\right], \tag{1.6.56}$$

where

$$R_y = \frac{\sqrt{2}c\sqrt{\cosh v\cosh y-\sinh v\sinh y\cos(\phi-\psi)-\cos(u-\beta)}}{\sqrt{\cosh v-\cos u}\,\sqrt{\cosh y-\cos\beta}}, \tag{1.6.57}$$

and

$$\chi(x) = \frac{2c\,|\cos[(u-\beta)/2]|\,\sqrt{\cosh x-\cosh y}\,\sqrt{\cosh x-\cosh t_1}}{\sinh x\sqrt{\cosh v-\cos u}\,\sqrt{\cosh y-\cos\beta}}$$

$$= \frac{2c\sqrt{\cosh t_2-\cosh v}\,\sqrt{\cosh x-\cosh y}}{\sinh t_2\sqrt{\cosh v-\cos u}\,\sqrt{\cosh y-\cos\beta}}$$

$$= \frac{2c\,|\sin[(u-\beta)/2]|\,\sqrt{\cosh x - \cosh y}}{\sqrt{\cosh t_2 - \cosh x}\,\sqrt{\cosh v - \cos u}\,\sqrt{\cosh y - \cos\beta}}. \tag{1.6.58}$$

Equivalence of the expressions (58) can be proven by using formulae from Appendix. The following identity is valid:

$$\frac{t_1'(x)\,\lambda\left(\dfrac{\tanh(y/2)\tanh(v/2)}{\tanh^2(t_2/2)}, \phi-\psi\right)}{\sqrt{\cosh v - \cosh t_1}} = -\frac{(\cosh x - \cosh y)^{3/2}}{\sinh x}\frac{\mathrm{d}F(x)}{\mathrm{d}x}. \tag{1.6.59}$$

Substitution of (59) in (55) and integration by parts yield

$$\mathcal{L}\left(\tanh\frac{y}{2}\right)\left[\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{t_1'(x)\sinh x\,\lambda\left(\dfrac{\tanh(v/2)}{\tanh^2(t_2/2)}, \phi-\psi\right)\mathrm{d}x}{\sqrt{\cosh x - \cosh y}\,\sqrt{\cosh v - \cosh t_1}}\right] = -\frac{1}{2}F(b)\sinh y$$

$$= -\frac{2c^3\,|\sin(u-\beta)|\sinh y}{(\cosh v - \cos u)^{3/2}(\cosh y - \cos\beta)^{3/2}}\frac{1}{R_y^3}\left[\frac{R_y}{\chi(b)} + \tan^{-1}\left(\frac{\chi(b)}{R_y}\right)\right], \tag{1.6.60}$$

While integrating by parts in (60), one should notice that substitution of the lower limit of integration y leads to the uncertainty of the type $\infty-\infty$ which has to be dealt with properly.

Now substitution of (60) in (52) allows us to rewrite it in the form

$$V(v,u,\phi) = \frac{c^3\,|\sin(u-\beta)|}{\pi^2(\cosh v - \cos u)}\int_0^{2\pi}\int_0^b \frac{1}{R_y^3}\left[\frac{R_y}{\chi(b)} + \tan^{-1}\left(\frac{\chi(b)}{R_y}\right)\right]\frac{V(y,\psi)\sinh y\,\mathrm{d}y\,\mathrm{d}\psi}{(\cosh y - \cos\beta)^2}. \tag{1.6.61}$$

The last formula is in agreement with the classical result of Hobson (1900).

Problem 2. The boundary conditions in this case take the form

$$\sigma = \sigma_0(v,\phi), \qquad \text{for } 0 \le v \le b_0, \qquad u = u_0, \qquad 0 \le \phi \le 2\pi. \tag{1.6.62}$$

$$V = 0, \qquad \text{for } 0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi, \tag{1.6.63}$$

Denote the surface of the first cap as S_0, and the surface of the second cap as S. Introduce the following points, with their toroidal coordinates: $M(v,u,\phi)$, $N(x,\beta,\psi)$, $N_0(v_0,u_0,\phi_0)$, and $K(v,\beta,\phi)$. The potential in the space can be presented again through the simple layer distributions

$$V(M)=\iint\limits_{S}\frac{\sigma(N)\,\mathrm{d}S}{R(M,N)}+\iint\limits_{S_0}\frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(M,N_0)}. \qquad (1.6.64)$$

We note that σ_0 in (64) is known from (62) while σ is not yet known. It can be found from the integral equation which results from substitution of the second boundary condition (63) in (64), namely,

$$0=\iint\limits_{S}\frac{\sigma(N)\,\mathrm{d}S}{R(K,N)}+\iint\limits_{S_0}\frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(K,N_0)}. \qquad (1.6.65)$$

By using the procedure similar to (38)–(41), we can rewrite (65) as

$$2c\sqrt{\cosh v-\cos\beta}\int_0^v\frac{\mathrm{d}\tau}{\sqrt{\cosh v-\cosh\tau}}\int_\tau^b\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh\tau}}$$

$$=-\iint\limits_{S_0}\frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(K,N_0)}. \qquad (1.6.66)$$

The general solution to (66) is given in (46), we just need to substitute the right-hand side. Assuming that the order of integration is interchangeable we need to compute first

$$J=\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y\frac{\sinh v\,\mathrm{d}v}{\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}\,\mathcal{L}\left(\tanh\frac{v}{2}\right)\left[\frac{1}{R(K,N_0)}\right]. \qquad (1.6.67)$$

Make use of the integral representation

$$\frac{1}{R(K,N_0)}=\frac{\sqrt{\cosh v_0-\cos u_0}\sqrt{\cosh v-\cos\beta}}{\pi c}$$

$$\times\int_0^{t_{10}(v)}\frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v_0/2)\tanh(v/2)},\phi-\phi_0\right)\mathrm{d}\tau}{\sqrt{\cosh v_0-\cosh\tau}\sqrt{\cosh v-\cosh\gamma_0}}. \qquad (1.6.68)$$

Here $\gamma_0=\gamma(\tau,\beta,v_0,u_0)$ as it is defined by (14). Substitution of (68) in (67) yields

$$J = \frac{\sqrt{\cosh v_0 - \cos u_0}}{\pi c} \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \frac{\sinh v \,\mathrm{d}v}{\sqrt{\cosh y - \cosh v}}$$

$$\times \int_0^{t_{10}(v)} \frac{\lambda\left(\frac{\tanh^2(\tau/2)}{\tanh(v_0/2)}, \phi - \phi_0\right)\mathrm{d}\tau}{\sqrt{\cosh v_0 - \cosh\tau}\,\sqrt{\cosh v - \cosh\gamma_0}}$$

$$= \frac{\sqrt{\cosh v_0 - \cos u_0}\, t_{10}'(y)}{c\sqrt{\cosh v_0 - \cosh t_{10}(y)}}\,\lambda\left(\frac{\tanh^2[t_{10}(y)/2]}{\tanh(v_0/2)}, \phi - \phi_0\right). \tag{1.6.69}$$

Here we used the identity (43). The next step is to compute

$$J_1 = \mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \frac{\sinh y \,\mathrm{d}y}{\sqrt{\cosh y - \cosh s}}\,\mathcal{L}\left(\coth^2\frac{y}{2}\right)\{J\}, \tag{1.6.70}$$

where J is defined by (69). The elementary simplification results in

$$J_1 = \frac{\sqrt{\cosh v_0 - \cos u_0}}{c}\,\mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \lambda\left(\frac{\tanh(v_0/2)}{\tanh^2[t_{20}(y)/2]}, \phi - \phi_0\right)$$

$$\times \frac{t_{10}'(y)\sinh y \,\mathrm{d}y}{\sqrt{\cosh y - \cosh s}\,\sqrt{\cosh v_0 - \cosh t_{10}(y)}}. \tag{1.6.71}$$

The integral (71) has already been computed in (60), so that we can write the solution of (66) in the form

$$\sigma(N) = -\iint_{S_0} G(N, N_0)\,\sigma(N_0)\,\mathrm{d}S_0, \tag{1.6.72}$$

where the Green's function G is defined by

$$G(N, N_0) = \frac{c\,|\sin(u_0 - \beta)|}{\pi^2(\cosh v_0 - \cos u_0)\,R^3(N, N_0)}\left[\frac{R(N, N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N, N_0)}\right)\right], \tag{1.6.73}$$

with

$$\chi_0(y) = \frac{2c\,\sqrt{\cosh t_{20}(y) - \cosh v_0}\,\sqrt{\cosh y - \cosh x}}{\sinh t_{20}(y)\,\sqrt{\cosh v_0 - \cos u_0}\,\sqrt{\cosh x - \cos\beta}}. \tag{1.6.74}$$

The back substitution of (72) in (64) allows us to express the potential in space directly in terms of the prescribed charge distribution σ_0. The integrals involved, though looking quite formidable, can be computed in terms of elementary functions. The main integral to be computed is

$$J_2 = \int\!\!\int_S \frac{1}{R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right]\frac{\mathrm{d}S_N}{R(M,N)}$$

$$= \frac{c}{\pi}\sqrt{\cosh v - \cos u}\int_0^{2\pi}\mathrm{d}\psi\int_0^b \frac{\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}}$$

$$\times\int_0^x \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]}{\tanh(v/2)\tanh(x/2)},\ \phi-\psi\right)\mathrm{d}t_1(y)}{\sqrt{\cosh v - \cosh t_1(y)}\,\sqrt{\cosh x - \cosh y}}$$

$$\times\frac{1}{R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right]. \tag{1.6.75}$$

Interchanging the order of integration and using the integral representation (60), we obtain

$$J_2 = -\frac{c}{\pi}\sqrt{\cosh v - \cos u}\int_0^{2\pi}\mathrm{d}\psi\int_0^b \frac{\mathrm{d}t_1(y)}{\sqrt{\cosh v - \cosh t_1(y)}}$$

$$\times\int_y^b \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]}{\tanh(v/2)\tanh(x/2)},\ \phi-\psi\right)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh y}}\left\{\frac{(\cosh x - \cos\beta)^{3/2}(\cosh v_0 - \cos u_0)^{3/2}}{2c^3\,|\sin(\beta - u_0)|\,\sinh x}\right.$$

$$\left.\times\,\mathcal{L}\left(\tanh\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{t_{10}'(s)\,\lambda\left(\dfrac{\tanh(v_0/2)}{\tanh^2[t_{20}(s)/2]},\ \psi-\phi_0\right)\sinh s\,\mathrm{d}s}{\sqrt{\cosh s - \cosh x}\,\sqrt{\cosh v_0 - \cosh t_{10}(s)}}\right\}. \tag{1.6.76}$$

Some obvious simplifications can be made, with the result

$$J_2 = -\frac{\sqrt{\cosh v - \cos u}\,(\cosh v_0 - \cos u_0)^{3/2}}{2\pi c^2\,|\sin(\beta - u_0)|}\int_0^{2\pi} \mathrm{d}\psi \int_0^b \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]}{\tanh(v/2)}, \phi - \psi\right)\mathrm{d}t_1(y)}{\sqrt{\cosh v - \cosh t_1(y)}}$$

$$\times \int_y^b \frac{\mathrm{d}x}{\sqrt{\cosh x - \cosh y}}\,\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{t_{10}'(s)\,\lambda\left(\dfrac{\tanh(v_0/2)}{\tanh^2[t_{20}(s)/2]}, \psi - \phi_0\right)\sinh s\,\mathrm{d}s}{\sqrt{\cosh s - \cosh x}\,\sqrt{\cosh v_0 - \cosh t_{10}(s)}}. \quad (1.6.77)$$

Further simplification is due to (48):

$$J_2 = \frac{\sqrt{\cosh v - \cos u}\,(\cosh v_0 - \cos u_0)^{3/2}}{2c^2\,|\sin(\beta - u_0)|}$$

$$\times \int_0^b \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]\,\tanh(v_0/2)}{\tanh^2[t_{20}(y)/2]\,\tanh(v/2)}, \phi - \phi_0\right)t_1'(y)\,t_{10}'(y)\,\mathrm{d}y}{\sqrt{\cosh v - \cosh t_1(y)}\,\sqrt{\cosh v_0 - \cosh t_{10}(y)}}. \quad (1.6.78)$$

The integral in (78) is similar to the one computed in (20). The final result is

$$\int\!\!\int_S \frac{1}{R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right]\frac{\mathrm{d}S_N}{R(M,N)}$$

$$= \frac{\sqrt{\cosh v - \cos u}\,(\cosh v_0 - \cos u_0)^{3/2}}{2c^2\,|\sin(\beta - u_0)|}$$

$$\times\left\{\frac{1}{\sqrt{2[\cosh w - \cos(u - u_0)]}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{2}\eta_1(b)}{\sqrt{\cosh w - \cos(u - u_0)}}\right)\right]\right.$$

$$\left. + \frac{1}{\sqrt{2[\cosh w - \cos(u + u_0 - 2\beta)]}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{2}\eta_2(b)}{\sqrt{\cosh w - \cos(u + u_0 - 2\beta)}}\right)\right]\right\}, \quad (1.6.79)$$

with $\eta_{1,2}$ defined by (21) and $\cosh w$ given by (26). Substitution of (79) in (64) allows us to express the potential in the form

$$V(M) = \int\int_{S_0} \left\{ 1 + \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\eta_1(b)}{\sqrt{\cosh w - \cos(u - u_0)}} \right) - \frac{\sqrt{\cosh w - \cos(u - u_0)}}{\sqrt{\cosh w - \cos(u + u_0 - 2\beta)}} \left[1 \right. \right.$$

$$\left. \left. - \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\eta_2(b)}{\sqrt{\cosh w - \cos(u + u_0 - 2\beta)}} \right) \right] \right\} \frac{\sigma_0(N_0)\, dS_0}{2R(M, N_0)} \quad (1.6.80)$$

In the particular case of $b \to \infty$ formula (80) simplifies as follows:

$$V(M) = \int\int_{S_0} \left\{ 1 + \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\cos[(u - u_0)/2]}{\sqrt{\cosh w - \cos(u - u_0)}} \right) - \frac{\sqrt{\cosh w - \cos(u - u_0)}}{\sqrt{\cosh w - \cos(u + u_0 - 2\beta)}} \left[1 \right. \right.$$

$$\left. \left. + \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\cos[(u + u_0 - 2\beta)/2]}{\sqrt{\cosh w - \cos(u + u_0 - 2\beta)}} \right) \right] \right\} \frac{\sigma_0(N_0)\, dS_0}{2R(M, N_0)} \quad (1.6.81)$$

The last formula is in agreement with the long standing result of Hobson (1900). Several examples are considered below.

Spherical cap charged to a uniform potential. Consider a spherical cap defined by $0 \le v \le b$, $u = \beta$, with a uniform potential V_0 prescribed at its surface. The charge distribution can be found from (46), and is

$$\sigma = \frac{V_0\sqrt{1 - \cos\beta}}{2\pi^2 c} \left\{ \left[\frac{(\cosh b + 1)(\cosh v - \cos\beta)}{\cosh b - \cosh v} \right]^{1/2} \right.$$

$$\left. + \sqrt{1 + \cos\beta}\, \tan^{-1}\left[\frac{(1 + \cos\beta)(\cosh b - \cosh v)}{(1 + \cosh b)(\cosh v - \cos\beta)} \right]^{1/2} \right\} \quad (1.6.82)$$

The potential in space is conveniently defined by (50). The final result is

$$V = \frac{\sqrt{2}\, V_0 |\sin(\beta/2)|\, \sqrt{\cosh v - \cos u}}{\pi\, |\cos[(u - \beta)/2]|\, (x_1^2 + x_2^2)} \left[\left(1 \right. \right.$$

$$\left. + \frac{x_1^2}{\sin^2(\beta/2)} \right)^{1/2} \sin^{-1}\left(\frac{\sinh[t_1(b)/2]\sqrt{x_1^2 + \sin^2(\beta/2)}}{x_1\sqrt{x_2^2 + \sinh^2[t_1(b)/2)]}} \right)$$

$$-\left(1-\frac{x_2^2}{\sin^2(\beta/2)}\right)^{1/2}\sin^{-1}\left(\frac{\sinh^2[t_1(b)/2)]\sqrt{\sin^2(\beta/2)-x_2^2}}{x_2\sqrt{x_1^2-\sinh^2[t_1(b)/2)]}}\right)\Bigg]. \tag{1.6.83}$$

Here

$$x_{1,2}^2=(m^2+n^2)^{1/2}\pm m,\qquad n=\frac{\sinh(v/2)\sin(\beta/2)}{\cos[(u-\beta)/2]},$$

$$m=\frac{\sinh^2(v/2)-\sin^2(\beta/2)+\sin^2[(u-\beta)/2]}{2\cos^2[(u-\beta)/2]}. \tag{1.6.84}$$

Application of the reciprocal theorem to (82) allows us to express the total charge on a spherical cap with an *arbitrary* potential $V(v,\phi)$ prescribed at its surface as follows:

$$Q=\frac{c\sqrt{1-\cos\beta}}{2\pi^2}\int_0^{2\pi}\int_0^{b}\left\{\left[\frac{(\cosh b+1)(\cosh v-\cos\beta)}{\cosh b-\cosh v}\right]^{1/2}\right.$$

$$\left.+\sqrt{1+\cos\beta}\tan^{-1}\left[\frac{(1+\cos\beta)(\cosh b-\cosh v)}{(1+\cosh b)(\cosh v-\cos\beta)}\right]^{1/2}\right\}\frac{V(v,\phi)\sinh v\,\mathrm{d}v\,\mathrm{d}\phi}{(\cosh v-\cos\beta)^2}. \tag{1.6.85}$$

The same result can be obtained by a direct integration of both sides of (46). Formulae (82) and (83) in the case of $b\to\infty$ take the form

$$\sigma=\frac{V_0\sqrt{1-\cos\beta}}{2\pi^2 c}\left\{\sqrt{\cosh v-\cos\beta}\;+\sqrt{1+\cos\beta}\tan^{-1}\left[\frac{1+\cos\beta}{\cosh v-\cos\beta}\right]^{1/2}\right\} \tag{1.6.86}$$

$$V=\frac{\sqrt{2}\,V_0|\sin(\beta/2)|\sqrt{\cosh v-\cos u}}{\pi|\cos[(u-\beta)/2]|\,(x_1^2+x_2^2)}\left[\left(1+\frac{x_1^2}{\sin^2(\beta/2)}\right)^{1/2}\cos^{-1}\left(\frac{x_2\sin[(u-\beta)/2]}{\sqrt{x_2^2+\sinh^2(v/2)}}\right)\right.$$

$$\left.-\left(1-\frac{x_2^2}{\sin^2(\beta/2)}\right)^{1/2}\cos^{-1}\left(\frac{x_1\sin[(u-\beta)/2]}{\sqrt{x_1^2-\sinh^2(v/2)}}\right)\right]. \tag{1.6.87}$$

Electrified spherical ring. Consider the Dirichlet problem for a spherical ring $b \le v < \infty$, with the following boundary condition prescribed at its surface:

$$V = V(v, \phi), \quad \text{for} \quad u = \beta, \quad b \le v < \infty, \quad 0 < \phi \le 2\pi. \tag{1.6.88}$$

By using the procedure similar to the one employed in Problem 1 above, we come to the following expression for the potential in space:

$$V(v,u,\phi) = 2c\sqrt{\cosh v - \cos u}\left\{\int_0^{t_1(b)} \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh\tau}}\right.$$

$$\times \int_b^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\gamma}} + \int_{t_1(b)}^{v} \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\left.\times \int_\gamma^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\gamma}}\right\}. \tag{1.6.89}$$

Substitution of the boundary condition (88) in (89) leads to the integral equation

$$V(v,\phi) = 2c\sqrt{\cosh v - \cos u}\left\{\int_0^{b} \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh\tau}}\right.$$

$$\times \int_b^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}} + \int_{b}^{v} \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\left.\times \int_\tau^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}\right\}. \tag{1.6.90}$$

Let us apply the operator

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y \frac{\mathcal{L}[\tanh(v/2)]\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}$$

to both sides of (90). The result is

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}$$

$$=\int_0^b \frac{\sqrt{\cosh b-\cosh\tau}\,\sinh y\,\mathrm{d}\tau}{\sqrt{\cosh y-\cosh b}\,(\cosh y-\cosh\tau)}$$

$$\times\int_b^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh\tau}}$$

$$+\pi\int_y^\infty \frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh y}}. \tag{1.6.91}$$

We introduce a new unknown

$$p(y,\phi)=\int_y^\infty \frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh y}}. \tag{1.6.92}$$

Expression (92) can be easily inverted, namely,

$$\sigma(x,\phi)=-\frac{(\cosh x-\cos\beta)^{3/2}}{\pi\sinh x}\,\mathcal{L}\left(\tanh\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{\mathcal{L}\left(\coth\dfrac{z}{2}\right)p(z,\phi)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z-\cosh x}}. \tag{1.6.93}$$

Substitution of (93) in (91) leads to the governing integral equation

$$p(y,\phi)+\frac{1}{\pi^2}\int_b^\infty\left\{\int_0^b \mathcal{L}\left(\frac{\tanh^2(\tau/2)}{\tanh(y/2)\tanh(z/2)}\right)\frac{(\cosh b-\cosh\tau)\,\mathrm{d}\tau}{(\cosh y-\cosh\tau)(\cosh z-\cosh\tau)}\right\}$$

$$\times\frac{\sinh y\, p(z,\phi)\sinh z\,\mathrm{d}z}{\sqrt{\cosh y-\cosh b}\sqrt{\cosh z-\cosh b}}$$

$$=\frac{1}{\pi}\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y\frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}. \tag{1.6.94}$$

The following rule of interchange of the order of integration was used:

$$\int_b^\infty f(x)\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty\frac{F(z)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z-\cosh x}}=-\int_b^\infty F(z)\,\mathrm{d}z\frac{\mathrm{d}}{\mathrm{d}z}\int_b^z\frac{f(x)\sinh x\,\mathrm{d}x}{\sqrt{\cosh z-\cosh x}}.$$

It is important to notice that the integral in curly brackets of (94) can be computed exactly in terms of elementary functions for any specific harmonic. For example, in the case of axial symmetry, equation (94) takes the form

$$p(y)+\frac{\sinh y}{\pi^2\sqrt{\cosh y-\cosh b}}\int_b^\infty\frac{K(y)-K(z)}{\cosh y-\cosh z}\frac{p(z)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z-\cosh b}}$$

$$=\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y\frac{V(v)\sinh v\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}. \tag{1.6.95}$$

Here

$$K(x)=\frac{\cosh x-\cosh b}{\sinh x}\ln\left(\frac{\sinh[(x+b)/2]}{\sinh[(x-b)/2]}\right). \tag{1.6.96}$$

If one is interested in the quantity of total charge Q only, it can be expressed directly through function p as follows:

$$Q=\frac{c^2}{\pi}\sqrt{\cosh b-\cos\beta}\int_0^{2\pi}\int_b^\infty\frac{p(y,\phi)\sinh y\,\mathrm{d}y\,\mathrm{d}\phi}{\sqrt{\cosh y-\cosh b}\,(\cosh y-\cos\beta)}. \tag{1.6.97}$$

A complete solution to the problem is beyond the scope of this book.

Interaction of several charged spherical caps. Consider n spherical caps $u=\beta_k$, $0<v\le b_k$, $k=1,2,\ldots,n$, with arbitrary potentials prescribed on their surfaces. The boundary conditions can be formulated in the form

$$V=V_k(v,\phi),\quad \text{for}\quad u=\beta_k,\quad 0\le v\le b_k,\quad 0\le\phi<2\pi,\quad k=1,2,\ldots,n. \tag{1.6.98}$$

Again, the procedure outlined in the solution of Problem 1 leads to the expression for the potential

$$V(v,u,\phi)=2c\sqrt{\cosh v-\cos u}\sum_{k=1}^{n}\int_0^{t_{1k}(b_k)}\frac{d\tau}{\sqrt{\cosh v-\cosh\tau}}$$

$$\times\int_{\gamma_k}^{b_k}\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma_k(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_k)^{3/2}\sqrt{\cosh x-\cosh\gamma_k}}. \tag{1.6.99}$$

Here σ_k is the yet unknown charge distribution over the surface of the k-th cap, and the following notation was introduced:

$$t_{1k}\equiv t_1(x,\beta_k,v,u),\quad t_{2k}\equiv t_2(x,\beta_k,v,u),\quad \gamma_k\equiv\gamma(\tau,\beta_k,v,u). \tag{1.6.100}$$

We recall that the notations $t_{1,2}$ and γ were first introduced by (15) and (14) respectively.

Substitution of the boundary conditions (98) in (99) leads to a system of n integral equations. We can single out, without loss of generality, the cap number one. The corresponding integral equation takes the form

$$V_1(v,\phi)=2c\sqrt{\cosh v-\cos\beta_1}\int_0^{v}\frac{d\tau}{\sqrt{\cosh v-\cosh\tau}}$$

$$\times\int_{\tau}^{b_1}\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma_1(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_1)^{3/2}\sqrt{\cosh x-\cosh\tau}}+\sum_{k=2}^{n}\int_0^{t_{1k1}(b_k)}\frac{d\tau}{\sqrt{\cosh v-\cosh\tau}}$$

$$\times\int_{\gamma_{k1}}^{b_k}\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma_k(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_k)^{3/2}\sqrt{\cosh x-\cosh\gamma_{k1}}}. \tag{1.6.101}$$

Here

$$\gamma_{kl}\equiv\gamma(\tau,\beta_k,v,\beta_l),\quad t_{1kl}\equiv t_1(x,\beta_k,u,\beta_l). \tag{1.6.102}$$

Application of the integral operator

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y\frac{\mathcal{L}[\tanh(v/2)]\sinh v\,dv}{2c\sqrt{\cosh v-\cos\beta_1}\sqrt{\cosh y-\cosh v}}$$

to both sides of (101) results in

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y\frac{\mathcal{L}[\tanh(v/2)]\,V_1(v,\phi)\sinh v\,dv}{2c\sqrt{\cosh v-\cos\beta_1}\sqrt{\cosh y-\cosh v}}$$

$$=\pi\int_y^{b_1}\frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_1)^{3/2}\sqrt{\cosh x-\cosh y}}$$

$$+\pi\sum_{k=2}^{n}\int_0^{b_k}\mathcal{L}\left(\frac{\tanh^2[t_1(x,\beta_k,y,\beta_1)/2]}{\tanh(y/2)\tanh(x/2)}\right)$$

$$\times\frac{[\partial t_1(x,\beta_k,y,\beta_1)/\partial y]\,\sigma_k(x,\phi)\sinh x\,dx}{\sqrt{\cosh x-\cosh t_1(x,\beta_k,y,\beta_1)}(\cosh x-\cos\beta_k)^{3/2}}. \tag{1.6.103}$$

Introduce a new variable q_k as follows

$$q_k(y,\phi)=\int_y^{b_k}\frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma_k(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_k)^{3/2}\sqrt{\cosh x-\cosh y}}. \tag{1.6.104}$$

Expression (104) can be easily inverted, namely,

$$\sigma_k(x,\phi) = -\frac{(\cosh x - \cos\beta_k)^{3/2}}{\pi\sinh x}\,\mathcal{L}\left(\tanh\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{b_k}\frac{\mathcal{L}\left(\coth\frac{z}{2}\right)q_k(z,\phi)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z - \cosh x}} \tag{1.6.105}$$

Substitution of (104) in (103) leads to the governing integral equation

$$q_1(y,\phi) + \frac{1}{\pi}\sum_{k=2}^{n}\int_0^{b_k}\left\{\mathcal{L}\left(\coth\frac{z}{2}\right)\frac{\mathrm{d}}{\mathrm{d}z}\int_0^z\frac{[\partial t_1(x,\beta_k,y,\beta_1)/\partial y]\sinh x\,\mathrm{d}x}{\sqrt{\cosh x - \cosh t_1(x,\beta_k,y,\beta_1)}\sqrt{\cosh z - \cosh x}}\right.$$

$$\left.\times\,\mathcal{L}\left(\frac{\tanh^2[t_1(x,\beta_k,y,\beta_1)/2]}{\tanh(y/2)}\right)\right\}q_k(z,\phi)\,\mathrm{d}z$$

$$= \frac{1}{\pi}\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y\frac{\mathcal{L}[\tanh(v/2)]\,V_1(v,\phi)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v - \cos\beta_1}\sqrt{\cosh y - \cosh v}} \tag{1.6.106}$$

We can note again that the integral in curly brackets in (106) can be computed in terms of elementary functions for any specific harmonic. For example, in the case of axial symmetry, equation (106) takes the form

$$q_1(y) + \sum_{k=2}^{n}\frac{2\,|\sin[(\beta_1-\beta_k)/2]|\cosh(y/2)}{\pi\cos^4[(\beta_1-\beta_k)/2]}$$

$$\times\int_0^{b_k}\frac{\{\cosh z + \cosh y - 2\cos^2[(\beta_1-\beta_k)/2]\}\cosh(z/2)\,q_k(z)\,\mathrm{d}z}{[\cosh t_2(z,\beta_k,y,\beta_1) - \cosh t_1(z,\beta_k,y,\beta_1)]^2}$$

$$= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y\frac{V_1(v)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v - \cos\beta_1}\,\sqrt{\cosh y - \cosh v}}. \tag{1.6.107}$$

Expression (107) can also be rewritten as

$$q_1(y) + \sum_{k=2}^{n} \frac{|\sin[(\beta_1-\beta_k)/2]|}{\pi} \int_0^{b_k} \left[\frac{\cosh[(y+z)/2]}{\cosh(y+z)-\cos(\beta_1-\beta_k)} \right.$$

$$\left. + \frac{\cosh[(y-z)/2]}{\cosh(y-z)-\cos(\beta_1-\beta_k)} \right] q_k(z)\,\mathrm{d}z$$

$$= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y} \int_0^{y} \frac{V_1(v)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta_1}\,\sqrt{\cosh y-\cosh v}}. \qquad (1.6.108)$$

The remaining $n-1$ integral equations can be derived in a similar manner. Equation (108) is in agreement with the result of Ufliand (1977) who obtained it by using the Mehler-Fok integral transform. The last formula simplifies when V_1 is a constant, namely,

$$q_1(y) + \sum_{k=2}^{n} \frac{|\sin[(\beta_1-\beta_k)/2]|}{\pi} \int_0^{b_k} \left[\frac{\cosh[(y+z)/2]}{\cosh(y+z)-\cos(\beta_1-\beta_k)} \right.$$

$$\left. + \frac{\cosh[(y-z)/2]}{\cosh(y-z)-\cos(\beta_1-\beta_k)} \right] q_k(z)\,\mathrm{d}z = \frac{V_1\cosh(y/2)\,|\sin(\beta_1/2)|}{\pi c(\cosh y-\cos\beta_1)} \qquad (1.6.109)$$

The total charge Q_k can be expressed directly through function q as follows:

$$Q_k = \frac{2}{\pi}c^2|\sin(\beta_k/2)| \int_0^{2\pi}\int_0^{b_k} \frac{q_k(v,\phi)\cosh(v/2)\,\mathrm{d}v\,\mathrm{d}\phi}{\cosh v-\cos\beta_k}. \qquad (1.6.110)$$

The reader is advised to try to obtain a complete solution to the problem.

Appendix. Some essential formulae used in the main body of this section are presented here.

$$\tanh\left(\frac{t_1}{2}\right)\tanh\left(\frac{t_2}{2}\right) = \tanh\left(\frac{x}{2}\right)\tanh\left(\frac{v}{2}\right), \qquad (1.6.\mathrm{A1})$$

$$(\cosh t_2-\cosh v)(\cosh v-\cosh t_1) = \sinh^2 v\,\tan^2\left(\frac{u-\beta}{2}\right), \qquad (1.6.\mathrm{A2})$$

$$\cos^2\left(\frac{u-\beta}{2}\right)(\cosh t_1-\cosh\tau)(\cosh t_2-\cosh\tau)=(\cosh v-\cosh\tau)[\cosh x-\cosh\gamma(\tau)], \quad (1.6.\text{A}3)$$

$$\cosh t_1\cosh t_2=\frac{\sin^2[(u-\beta)/2]+\cosh v\ \cosh x}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}4)$$

$$\cosh t_1+\cosh t_2=\frac{\cosh v+\cosh x}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}5)$$

$$(\cosh x-\cosh t_1)(\cosh v-\cosh t_1)=\sin^2\left(\frac{u-\beta}{2}\right)\sinh^2 t_1, \quad (1.6.\text{A}6)$$

$$(\cosh t_2-\cosh x)(\cosh t_2-\cosh v)=\sin^2\left(\frac{u-\beta}{2}\right)\sinh^2 t_2, \quad (1.6.\text{A}7)$$

$$\sinh t_1\sinh t_2=\frac{\sinh x\sinh v}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}8)$$

$$\frac{\sinh t_1}{\sinh t_2}=\frac{\sinh v(\cosh x-\cosh t_1)}{\sinh x(\cosh t_2-\cosh v)}=\frac{\sinh x(\cosh v-\cosh t_1)}{\sinh v(\cosh t_2-\cosh x)}, \quad (1.6.\text{A}9)$$

$$(\cosh t_1-1)(\cosh t_2-1)=\frac{(\cosh v-1)(\cosh x-1)}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}10)$$

$$\cosh t_2-\cosh t_1=\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)}\sqrt{\cosh(x-v)-\cos(u-\beta)}}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}11)$$

$$\frac{\partial t_1}{\partial x}=\frac{\sqrt{\cosh v-\cosh t_1}\sqrt{\cosh t_2-\cosh x}}{|\cos[(u-\beta)/2]|\,(\cosh t_2-\cosh t_1)}=\frac{\sinh x(\cosh v-\cosh t_1)}{\sinh t_1(\cosh t_2-\cosh t_1)\cos^2[(u-\beta)/2]}$$

$$=\frac{|\sin[(u-\beta)/2]|\sinh x\sqrt{\cosh v-\cosh t_1}}{\cos^2[(u-\beta)/2]\,(\cosh t_2-\cosh t_1)\sqrt{\cosh x-\cosh t_1}}, \quad (1.6.\text{A}12)$$

$$\frac{\partial t_2}{\partial x}=\frac{\sinh x(\cosh t_2-\cosh v)}{\cos^2[(u-\beta)/2]\sinh t_2(\cosh t_2-\cosh t_1)}=\frac{\sqrt{\cosh x-\cosh t_1}\sqrt{\cosh t_2-\cosh v}}{|\cos[(u-\beta)/2]|\,(\cosh t_2-\cosh t_1)}$$

$$=\frac{|\sin[(u-\beta)/2]|\sinh x\sqrt{\cosh t_2-\cosh v}}{\cos^2[(u-\beta)/2]\,(\cosh t_2-\cosh t_1)\sqrt{\cosh t_2-\cosh x}}, \quad (1.6.\text{A}13)$$

$$\frac{t_2'}{t_1'}=\frac{\sqrt{\cosh x-\cosh t_1}\sqrt{\cosh t_2-\cosh v}}{\sqrt{\cosh v-\cosh t_1}\sqrt{\cosh t_2-\cosh x}}=\frac{\sinh v(\cosh x-\cosh t_1)}{\sinh x(\cosh v-\cosh t_1)}$$

$$=\frac{\sinh x(\cosh t_2-\cosh v)}{\sinh v(\cosh t_2-\cosh x)}=\frac{\sinh t_1}{\sinh t_2}\sinh^2 v\tan^2[(u-\beta)/2], \qquad (1.6.A14)$$

Exercises 1

1. Prove the identity $\mathcal{L}(p)\lambda(q,\phi)=\lambda(pq,\phi)$, for $p<1$ and $q<1$.
Hint: use (1.1.5)

2. Evaluate the integral

$$\int_0^{2\pi}\frac{d\phi}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)][r^2+r_0^2-2rr_0\cos(\phi-\psi)]}, \quad \text{for } \rho>\rho_0 \text{ and } r>r_0.$$

Answer: $\dfrac{2\pi(\rho^2r^2-\rho_0^2r_0^2)}{(\rho^2-\rho_0^2)(r^2-r_0^2)[\rho^2r^2+\rho_0^2r_0^2-2\rho\rho_0rr_0\cos(\phi_0-\psi)]}$.

3. Evaluate the integral

$$\int_0^{2\pi}\frac{d\phi}{[1+k^2-2k\cos(\phi-\phi_0)]^2[1+k_1^2-2k_1\cos(\phi-\psi)]}, \quad \text{for } k<1 \text{ and } k_1<1.$$

Answer: $\dfrac{2\pi}{1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)}\left\{\dfrac{2k^2}{(1-k^2)^3}\right.$

$$\left.+\frac{1}{1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)}\left[\frac{k_1^2}{1-k_1^2}+\frac{1-k^4k_1^2}{(1-k^2)^2}\right]\right\}.$$

4. Evaluate the integral

$$\int_0^{2\pi}\frac{e^{i\phi}\,d\phi}{[1+k^2-2k\cos(\phi-\phi_0)]^2[1+k_1^2-2k_1\cos(\phi-\psi)]}, \quad \text{for } k<1 \text{ and } k_1<1.$$

Answer: $\dfrac{2\pi}{[1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)]^2}\left\{\dfrac{k_1^3e^{i\psi}}{1-k_1^2}\right.$

$$+\frac{k\mathrm{e}^{i\phi_0}[2(1+k^4k_1^2)-kk_1(1+k^2)\mathrm{e}^{-i(\psi-\phi_0)}]+k_1\mathrm{e}^{i\psi}(1-3k^2)}{(1-k^2)^3}\Bigg\}.$$

5. Evaluate the integral

$$\int_0^{2\pi}\frac{\mathrm{e}^{2i\phi}\,\mathrm{d}\phi}{[1+k^2-2k\cos(\phi-\phi_0)]^2[1+k_1^2-2k_1\cos(\phi-\psi)]},\quad \text{for } k<1 \text{ and } k_1<1.$$

Answer: $$\frac{2\pi}{[1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)]^2}\Bigg\{\frac{k_1^2e^{2i\psi}}{1-k_1^2}$$

$$+\frac{2kk_1\mathrm{e}^{-i(\psi-\phi_0)}[e^{2i\psi}(k^4-3k^2+1)-k^2\mathrm{e}^{2i\phi_0}]-k^2\mathrm{e}^{2i\phi_0}(k^2+k_1^2-3-3k^2k_1^2)}{(1-k^2)^3}\Bigg\}.$$

6. Prove the identity (η is defined by 1.2.2)

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=-\frac{\rho^2\rho_0^2-x^4}{x^3\eta}.$$

7. Prove the identity

$$\lambda\Big(\frac{x^2}{\rho\rho_0},\phi-\phi_0\Big)=-\frac{x\eta}{R^2+\eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x},\quad \text{where } R=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}.$$

Hint: use the identity: $x^2+\rho^2\rho_0^2/x^2-2\rho\rho_0\cos(\phi-\phi_0)=R^2+\eta^2$, and the result above.

8. Let $\eta=\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}/x$. Prove the identity

$$\lambda\Big(\frac{\rho\rho_0}{x^2},\phi-\phi_0\Big)=\frac{x\eta}{R^2+\eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x}.$$

9. Prove the identities

$$\sqrt{l_2^2-\rho^2}\sqrt{l_2^2-a^2}=zl_2,\quad \sqrt{a^2-l_1^2}(\rho^2-l_1^2)^{1/2}=zl_1,$$

$$\sqrt{a^2-l_1^2}\sqrt{l_2^2-a^2}=za,\quad \sqrt{l_2^2-\rho^2}(\rho^2-l_1^2)^{1/2}=z\rho.$$

Reminder: l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively.
Hint: use (1.2.6)

10. Prove that $g(x)$ is inverse to both l_1 and l_2, namely, prove that $g(l_1)=a$, and $g(l_2)=a$.

11. Prove the identities

$$\frac{\partial l_1}{\partial z}=-\frac{zl_1}{l_2^2-l_1^2},\quad \frac{\partial l_2}{\partial z}=\frac{zl_2}{l_2^2-l_1^2},$$

$$\frac{\partial l_1}{\partial \rho}=\frac{al_2-\rho l_1}{l_2^2-l_1^2}=\frac{\rho(a^2-l_1^2)}{l_1(l_2^2-l_1^2)},\quad \frac{\partial l_2}{\partial \rho}=\frac{\rho l_2-al_1}{l_2^2-l_1^2}=\frac{\rho(l_2^2-a^2)}{l_2(l_2^2-l_1^2)}.$$

Hint: use the properties above.

12. Evaluate the integral

$$\int\frac{\mathrm{d}x}{\sqrt{\rho_0^2-x^2}}\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\lambda\left(\frac{l_1(x)x}{l_2(x)\rho_0},\phi-\phi_0\right)$$

Answer: $-\dfrac{1}{R_0}\tan^{-1}\dfrac{\sqrt{\rho_0^2-x^2}\sqrt{l_2^2(x)-x^2}}{xR_0}$.

Hint: use (1.2.21)

13. Evaluate the integral

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\frac{(x^2-l_1^2(x))^{1/2}}{[l_2^2(x)-l_1^2(x)]}\lambda\left(\frac{\rho\rho_0}{l_2^2(x)},\phi-\phi_0\right)$$

Answer: $\dfrac{1}{R_0}\tan^{-1}\dfrac{(x^2-l_1^2(x))^{1/2}\sqrt{x^2-\rho_0^2}}{xR_0}$.

Hint: use (1.2.15)

14. Establish the integral representation

$$\frac{2}{\pi}\int_0^{min(\rho_0,\rho)}\frac{\cos[\kappa\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}/x+(\pi\nu/2)]}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+\nu)/2}}\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)x^{\nu}\mathrm{d}x=\frac{e^{-\kappa R}}{R^{1+\nu}}.$$

15. Prove that h in (1.3.35) can be defined by any of the expressions

$$h\equiv h(a)=\frac{\sqrt{a^2-\rho_0^2}\sqrt{a^2-l_1^2}}{a}=\frac{z\sqrt{a^2-\rho_0^2}}{\sqrt{l_2^2-a^2}}$$

$$= \frac{[l_2^2 - l_1^2(\rho_0)]^{1/2}[l_2^2 - l_2^2(\rho_0)]^{1/2}}{l_2} = \frac{\sqrt{a^2-\rho_0^2}\sqrt{l_2^2-\rho^2}}{l_2}.$$

16. Prove that j in (1.4.21) can be defined in several equivalent ways:

$$j \equiv j(a) = \frac{\sqrt{\rho_0^2 - a^2}\sqrt{l_2^2 - a^2}}{a} = \frac{\sqrt{\rho_0^2-a^2}\sqrt{\rho^2-l_1^2}}{l_1}$$

$$= \frac{z\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-l_1^2}} = \frac{[l_1^2(\rho_0) - l_1^2]^{1/2}[l_2^2(\rho_0)-l_1^2]^{1/2}}{l_1}.$$

17. A circular conducting disc is kept at the potential v_0. Find the potential function V.

Answer: $V(\rho,z) = \frac{2}{\pi} v_0 \sin^{-1}(l_1/\rho) = \frac{2}{\pi} v_0 \sin^{-1}(\frac{a}{l_2})$.

Hint: use formula (1.3.23)

18. Subject to the conditions of the previous problem, find the charge distribution σ by using formulae (1.3.15) and (1.3.5). Prove that in both cases the result is the same.

Answer: $\sigma = \frac{v_0}{\pi^2(a^2-\rho^2)^{1/2}}$.

19. Solve problems 17 and 18 for $v = v_1\rho\cos\phi$, $v_1 = \text{const}$.

Answer: $V(\rho,\phi,z) = \frac{2}{\pi}\rho v_1 \cos\phi\left[\sin^{-1}(\frac{a}{l_2}) - \frac{a}{l_2}\sqrt{1-(a/l_2)^2}\right]$,

$$\sigma(\rho,\phi) = \frac{2v_1\rho\cos\phi}{\pi^2(a^2-\rho^2)^{1/2}}.$$

20. A uniform charge density $\sigma = \sigma_0 = \text{const}$ is prescribed over a circular disc of radius a, and potential $v = 0$ for $\rho > a$. Find the potential function.

Answer: $V(\rho,z) = 4\sigma_0\left[\sqrt{a^2-l_1^2} - z\sin^{-1}(\frac{a}{l_2})\right]$.

21. Solve the previous problem for the case where $\sigma = \sigma_1\rho\cos\phi$, $\sigma_1 = \text{const}$.

Answer: $V(\rho,\phi,z) = \frac{8}{3}\sigma_1\rho\cos\phi\left[\sqrt{a^2 - l_1^2}\left(\frac{3}{2} - \frac{a^2}{2l_2^2}\right) - \frac{3}{2}z\sin^{-1}(\frac{a}{l_2})\right].$

22. The potential function is given by the expression

$$V(\rho,\phi,z) = \frac{8}{3}\sigma_1\rho\cos\phi\left[\sqrt{a^2 - l_1^2}\left(\frac{3}{2} - \frac{a^2}{2l_2^2}\right) - \frac{3}{2}z\sin^{-1}(\frac{a}{l_2})\right].$$

Find the charge distribution on the plane $z = 0$.
Answer: $\sigma = \sigma_1\rho\cos\phi$, for $\rho \le a$;

$$\sigma = -\frac{4}{3\pi}\sigma_1\rho\cos\phi\left[\frac{a}{\sqrt{\rho^2 - a^2}}\left(\frac{3}{2} - \frac{a^2}{2\rho^2}\right) - \frac{3}{2}\sin^{-1}\left(\frac{a}{\rho}\right)\right]. \quad , \quad \text{for} \quad \rho > a.$$

Hint: use the second formula of (1.3.5).

23. The following boundary conditions are prescribed at $z = 0$

$$V = v_0/\rho, \quad \text{for } \rho \ge a, \ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, \ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{2v_0}{\pi(\rho^2 + z^2)^{1/2}}\sin^{-1}\left(\frac{(\rho^2 + z^2)^{1/2}}{l_2}\right),$

$$\sigma(\rho,\phi) = \frac{v_0 a}{\pi^2\rho^2\sqrt{\rho^2 - a^2}}.$$

The total charge is equal v_0.

24. The following boundary conditions are prescribed at $z = 0$

$$V = v_0/\rho^2, \quad \text{for } \rho \ge a, \ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, \ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{v_0}{\rho^2 + z^2}\left[1 - \frac{\sqrt{l_2^2 - \rho^2}}{l_2}\right.$

$$+\frac{z}{(\rho^2+z^2)^{1/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]}\Bigg],$$

$$\sigma(\rho,\phi)=\frac{v_0}{2\pi\rho^2}\left[\frac{1}{\sqrt{\rho^2-a^2}}-\frac{1}{\rho}\ln\frac{\rho+\sqrt{\rho^2-a^2}}{a}\right].$$

25. The following boundary conditions are prescribed at $z=0$

$$V=v_0/\rho^3,\quad \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z)=\dfrac{4v_0}{(\rho^2+z^2)^2}\left[\dfrac{z^2}{\sqrt{a^2-l_1^2}}-\dfrac{l_1^2\sqrt{a^2-l_1^2}}{2a^2}\right.$

$$\left.+\frac{\rho^2-2z^2}{2(\rho^2+z^2)^{1/2}}\sin^{-1}\left(\frac{(\rho^2+z^2)^{1/2}}{l_2}\right)\right].$$

Note that the potential at the coordinate origin is finite, namely, $V(0,0,0)=4v_0/(3\pi a^3)$.

$$\sigma(\rho,\phi)=\frac{2v_0(2a^2-\rho^2)}{\pi^2 a\rho^4\sqrt{\rho^2-a^2}}.$$

26. The following boundary conditions are prescribed at $z=0$

$$V=v_0/\rho^4,\quad \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z)=\dfrac{3v_0}{2(\rho^2+z^2)^2}\left\{\dfrac{\rho^2-z^2}{\rho^2+z^2}\left[1-\dfrac{\sqrt{l_2^2-\rho^2}}{l_2}\right]\right.$

$$-\frac{1}{3}\left[1-\left(\frac{\sqrt{l_2^2-\rho^2}}{l_2}\right)^3\right]+\frac{z}{2(\rho^2+z^2)}\left[\frac{l_2^2}{a^2}\sqrt{l_2^2-a^2}-z\right]$$

$$-\frac{z(2z^2-3\rho^2)}{2(\rho^2+z^2)^{3/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]},$$

The last expression simplifies at $z=0$:

$$V(\rho,\phi,0)=\frac{v_0}{\rho^4}\left[1-\frac{\sqrt{a^2-\rho^2}}{a}-\frac{\rho^2\sqrt{a^2-\rho^2}}{2a^3}\right],\ \text{for } \rho\le a;$$

and $V(\rho,\phi,0)=v_0/\rho^4$, for $\rho>a$. Note that the potential at the coordinate origin is finite, namely, $V(0,0,0)=3v_0/(8a^4)$.

$$\sigma(\rho,\phi)=\frac{3v_0}{8\pi\rho^4}\left\{\frac{3a^2-\rho^2}{a^2\sqrt{\rho^2-a^2}}-\frac{3}{\rho}\ln\left[\frac{\rho+\sqrt{\rho^2-a^2}}{a}\right]\right\}.$$

27. The following boundary conditions are prescribed at $z=0$

$$V=(v_1/\rho)\,e^{i\phi},\ \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\qquad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function.

Answer: $V(\rho,\phi,z)=\frac{v_1}{\rho}e^{i\phi}\left[1-\frac{\sqrt{a^2-l_1^2}}{a}\right]$

28. The following boundary conditions are prescribed at $z=0$

$$V=(v_2/\rho^2)\,e^{2i\phi},\ \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function.

Answer: $V(\rho,\phi,z)=\frac{3v_2}{2\rho^2}e^{2i\phi}\left\{\left[1-\frac{\sqrt{a^2-l_1^2}}{a}\right]-\frac{1}{3}\left[1-\left(\frac{\sqrt{a^2-l_1^2}}{a}\right)^3\right]\right\}.$

29. The following boundary conditions are prescribed at $z=0$

$$V=(v_3/\rho^3)\,e^{3i\phi},\ \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, \, 0 \le \phi < 2\pi.$$

Find the potential function.

Answer: $$V(\rho,\phi,z) = \frac{15 v_3}{4\rho^3} e^{3i\phi} \left\{ \frac{1}{2}\left[1 - \frac{\sqrt{a^2 - l_1^2}}{a}\right] - \frac{1}{3}\left[1 - \left(\frac{\sqrt{a^2 - l_1^2}}{a}\right)^3\right] + \frac{1}{10}\left[1 - \left(\frac{\sqrt{a^2 - l_1^2}}{a}\right)^5\right]\right\}.$$

30. Let the following boundary conditions be prescribed at $z = 0$:

$$V = 0, \quad \text{for } \rho \le a, \; 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^2, \; \sigma_0 = \text{const}, \quad \text{for } \rho > a, \; 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{2\pi\sigma_0}{(\rho^2 + z^2)^{1/2}} \ln \frac{l_2[(\rho^2 + z^2)^{1/2} + \sqrt{l_2^2 - a^2}]}{a[(\rho^2 + z^2)^{1/2} + z]},$$

$$\sigma(\rho,\phi) = \frac{\sigma_0}{\rho^2} \Re\left[1 - \frac{a}{\sqrt{a^2 - \rho^2}}\right], \quad \sigma(0,0) = -\frac{\sigma_0}{2a^2}.$$

31. Let the following boundary conditions be prescribed at $z = 0$:

$$V = 0, \quad \text{for } \rho \le a, \; 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^3, \; \sigma_0 = \text{const}, \quad \text{for } \rho > a, \; 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{4\sigma_0}{\rho^2 + z^2}\left[\frac{\sqrt{l_2^2 - a^2}}{a} - \frac{z}{(\rho^2 + z^2)^{1/2}} \sin^{-1}\left(\frac{(\rho^2 + z^2)^{1/2}}{l_2}\right)\right],$$

$$\sigma(\rho,\phi) = \frac{2\sigma_0}{\pi\rho^2}\left[\frac{1}{\rho} \sin^{-1}\left(\frac{\rho}{a}\right) - \frac{1}{\sqrt{a^2 - \rho^2}}\right], \text{ for } \rho < a; \; \sigma(0,0) = -2\sigma_0/(3\pi a^3).$$

32. Let the following boundary conditions be prescribed at $z = 0$:

$$V = 0, \quad \text{for } \rho \le a, \; 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^4, \sigma_0 = \text{const}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \dfrac{\pi\sigma_0}{2(\rho^2+z^2)^2}\left\{\dfrac{2z}{a}\sqrt{a^2-l_1^2} \ -3z + \dfrac{l_2^2}{a^2}\sqrt{l_2^2-a^2}\right.$

$$\left.\frac{\rho^2-2z^2}{(\rho^2+z^2)^{1/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]}\right\},$$

$$\sigma(\rho,\phi) = \frac{\sigma_0}{\rho^4}\Re\left[1-\frac{2a^2-\rho^2}{2a\sqrt{a^2-\rho^2}}\right], \quad \sigma(0,0) = -\frac{\sigma_0}{8a^4}.$$

33. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_1/\rho)e^{i\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = 2\pi(\sigma_1/\rho)\, e^{i\phi}[\sqrt{l_2^2-a^2}-z]$,

$$\sigma(\rho,\phi) = (\sigma_1/\rho)e^{i\phi}\Re[1-a/\sqrt{a^2-\rho^2}].$$

34. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_2/\rho^2)e^{2i\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \pi(\sigma_2/\rho^2)\, e^{2i\phi}[\sqrt{l_2^2-a^2}-2z+z\sqrt{a^2-l_1^2}/a]$,

$$\sigma(\rho,\phi) = (\sigma_2/\rho^2)e^{i\phi}\Re\left[1-\frac{2a^2-\rho^2}{2a\sqrt{a^2-\rho^2}}\right].$$

35. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_3/\rho^3)e^{3i\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{3\pi\sigma_3}{4\rho^3} e^{3i\phi}\left[\sqrt{l_2^2 - a^2} - \frac{8}{3}z\right.$

$$\left. + 2\frac{z}{a}\sqrt{a^2 - l_1^2} - \frac{1}{3}z\left(\frac{\sqrt{a^2 - l_1^2}}{a}\right)^3\right],$$

$$\sigma(\rho,\phi) = \frac{\sigma_3}{\rho^3} e^{3i\phi}\Re\left[1 - \frac{3a}{8\sqrt{a^2-\rho^2}} - \frac{3\sqrt{a^2-\rho^2}}{4a} + \frac{(a^2-\rho^2)^{3/2}}{8a^3}\right].$$

36. Prove that the total charge Q_T in Problem 2 (1.4.26) can be expressed directly in terms of the given charge density σ as

$$Q_T = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty} \sigma(\rho,\phi)\cos^{-1}\left(\frac{a}{\rho}\right)\rho d\rho d\phi.$$

Hint: integrate (1.4.29).

37. Solve the problem above in the case when $\sigma = \sigma_0/\rho^n$.

Answer: $Q_T = \dfrac{2\sigma_0\sqrt{\pi}\,\Gamma[(n-1)/2]}{(n-2)\,\Gamma(n/2)a^{n-2}}.$

38. Prove that parameter χ given in (1.5.35) can be defined by an alternative expression

$$\chi = \frac{\sqrt{2}\sqrt{m_2^2(\alpha) - \sin^2(\alpha/2)\, m_2^2(\pi)}\sqrt{\cos\alpha - \cos\theta_0}}{\sin\alpha},$$

with m_1 and m_2 defined by (1.5.1).

39. Prove that in the limiting case $r \to a$ formula (1.5.34) takes the form

$$V(a,\theta,\phi) = \frac{\sqrt{\cos\theta - \cos\alpha}}{2\pi^2}\int_0^{2\pi} d\phi_0$$

$$\times\int_{\alpha}^{\pi}\frac{v(\theta_0,\phi_0)\sin\theta_0\,d\theta_0}{\sqrt{\cos\alpha-\cos\theta_0}\,[1-\cos\theta\cos\theta_0-\sin\theta\sin\theta_0\cos(\phi-\phi_0)]}.$$

40. Prove that in the case of axial symmetry formula in Example 39 above simplifies as follows:

$$V(a,\theta)=\frac{\sqrt{\cos\theta-\cos\alpha}}{\pi}\int_{\alpha}^{\pi}\frac{v(\theta_0)\sin\theta_0\,d\theta_0}{\sqrt{\cos\alpha-\cos\theta_0}\,(\cos\theta-\cos\theta_0)}.$$

41. Find the charge density distribution on a spherical cap $\alpha\le\theta\le\pi$ kept at a constant potential v_0.

Answer: $\sigma(\theta)=\dfrac{v_0}{2\pi^2 a}\left[\dfrac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha-\cos\theta}}+\tan^{-1}\dfrac{\sqrt{\cos\alpha-\cos\theta}}{\sqrt{1-\cos\alpha}}\right].$

42. Consider a mixed boundary value problem for a sphere, subject to the boundary conditions at $r=a$

$$\sigma(\theta,\phi)=q_0\cos\theta,\ \text{ for }\ 0\le\theta<\alpha,\ \ 0\le\phi<2\pi;$$

$$V(a,\theta,\phi)=v_0\sin\theta\cos\phi,\ \text{ for }\ \alpha\le\theta\le\pi,\ \ 0\le\phi<2\pi,$$

with q_0 and v_0 being constant. Find: *a*) charge density distribution for $\alpha\le\theta\le\pi$; *b*) the total charge on the sphere; *c*) the potential in space and on the sphere for $\theta<\alpha$.

Answers:

a) $$\sigma(\theta,\phi)=\frac{2}{\pi}q_0\cos\theta\left[-\left(1+\frac{1-\cos\alpha}{3\cos\theta}\right)\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha-\cos\theta}}+\tan^{-1}\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha-\cos\theta}}\right]$$

$$+\frac{v_0\sin\theta\cos\phi}{2\pi^2 a}\left[\mu_1\frac{3-\mu_1^2}{\sqrt{1-\mu_1^2}}+3\cos^{-1}\mu_1\right],$$

with $\mu_1=\sin(\alpha/2)/\sin(\theta/2)$, for $\theta>\alpha$;

b) $$Q=\frac{4}{3}q_0 a^2\sin\alpha(1-\cos\alpha);$$

c) $$V(r,\theta,\phi)=\frac{4q_0}{3r^2}\Big\{[(r^3+a^3)\cos^{-1}A_1-|r^3-a^3|\cos^{-1}A_2]\cos\theta$$

$$+2arB\left(A_1^2\cos^2\frac{\theta}{2}-A_2^2\sin^2\frac{\theta}{2}\right)\Big\}$$

$$+\frac{v_0\sin\theta\cos\phi}{\pi ar^2}[(r^3+a^3)\sin^{-1}A_1-|r^3-a^3|\sin^{-1}A_2-arB(A_1^2+A_2^2)],$$

where

$$A_1=\frac{(r+a)\cos(\alpha/2)}{\sqrt{m_2^2(\alpha)+4ar\cos^2(\theta/2)\cos^2(\alpha/2)}},$$

$$A_2=\frac{|r-a|\cos(\alpha/2)}{\sqrt{m_2^2(\alpha)-4ar\sin^2(\theta/2)\cos^2(\alpha/2)}},$$

$$B=\sqrt{m_2^2(\alpha)/\cos^2(\alpha/2)-m_2^2(0)}.$$

On the surface of the sphere

$$V(a,\theta,\phi)=\frac{8}{3}aq_0\left[\cos\theta\cos^{-1}\mu_2+2\cos\frac{\alpha}{2}\cos\frac{\theta}{2}\sqrt{1-\mu_2^2}\right]$$

$$+\frac{2}{\pi}v_0\sin\theta\cos\phi[\sin^{-1}\mu_2-\mu_2\sqrt{1-\mu_2^2}],$$

with $\mu_2=\cos(\alpha/2)/\cos(\theta/2)$, for $\theta<\alpha$

43. Prove that χ_0 in (1.5.59) can also be presented as

$$\chi_0=\frac{\sqrt{2}\sqrt{m_{20}^2(\alpha)-\sin^2(\alpha/2)\,m_{20}^2(\pi)}\sqrt{\cos\alpha-\cos\theta}}{\sin\alpha}.$$

44. Prove that $\eta_{1,2}$ in (1.5.68) can also be presented as

$$\eta_{1,2}(x)=\frac{T(x)}{2a}\pm\frac{(r^2-a^2)(r_0^2-a^2)}{2aT(x)},$$

with

$$T(x) = 2\sqrt{m_2^2(x) - \cos^2(x/2)\, m_2^2(0)}\,\sqrt{m_{20}^2(x) - \cos^2(x/2)\, m_{20}^2(0)}/\sin x.$$

45. Consider a boundary value problem for a charged sphere of radius a, with discontinuous boundary conditions at $r = a$

$$V(a,\theta,\phi) = v_0 = \text{const.}, \quad \text{for} \quad 0 \le \theta < \alpha, \ 0 \le \phi < 2\pi;$$

$$V(a,\theta,\phi) = 0, \quad \text{for} \quad \alpha \le \theta < \pi, \ 0 \le \phi < 2\pi;$$

Find a) charge density distribution, b) the total charge, c) potential due to the charged sphere.

Answers:

a)

$$q(\theta) = \frac{v_0}{2\pi^2 a}\left\{\frac{\Pi(-\kappa_1^2,\kappa_2) - K(\kappa_2)}{\sin(\alpha/2)\cos(\theta/2)} + \frac{2\sin(\alpha/2)\cos(\theta/2)}{\cos\theta - \cos\alpha}E(\kappa_2)\right\}, \quad \text{for} \ \ \theta < \alpha;$$

$$q(\theta) = \frac{v_0}{2\pi^2 a}\left\{\frac{\Pi(-\kappa_0^2,\kappa_2^{-1})}{\cos(\alpha/2)\sin(\theta/2)} + \frac{2\cos(\alpha/2)\sin(\theta/2)}{\cos\theta - \cos\alpha}E(\kappa_2^{-1})\right\},$$

$$\text{for} \ \ \pi > \theta > \alpha,$$

with $\kappa_0 = \tan(\alpha/2)$, $\kappa_1 = \tan(\theta/2)$, $\kappa_2 = \kappa_1/\kappa_0$; K, E, Π are the complete elliptic integrals of the first, second and third kind respectively.

b) $\quad Q = v_0 a \sin^2(\alpha/2);$

c) $$V(r,\theta,\phi) = \frac{2v_0}{\pi r}\frac{|r-a|}{\sqrt{1-c_1}\sqrt{1+c_3}}[\Pi(\kappa_3^2,\kappa_4) - \Pi(-\kappa_5^2,\kappa_4)],$$

where

$$c_1 = \cos t_2(\alpha), \quad c_3 = \cos t_1(\alpha), \quad \kappa_3 = \tan(\alpha/2)/\tan[t_2(\alpha)/2],$$

$$\kappa_4 = \tan[t_1(\alpha)/2]/\tan[t_2(\alpha)/2], \quad \kappa_5 = \tan[t_1(\alpha)/2].$$

Hint: for details see (Fabrikant, 1987a,e)

46. Prove the identities

$$[\cos\theta - \cos t_2(x)][\cos t_1(x) - \cos\theta] = \left(\frac{a-r}{a+r}\right)^2 \sin^2\theta,$$

$$[\cos x - \cos t_2(x)][\cos t_1(x) - \cos x] = \left(\frac{a-r}{a+r}\right)^2 \sin^2 x,$$

$$\frac{\partial t_2(x)}{\partial x} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos t_1(x) - \cos\theta]}\frac{\partial t_1(x)}{\partial x} = \frac{\sin x\,[\cos\theta - \cos t_2(x)]}{\sin\theta\,[\cos x - \cos t_2(x)]}\frac{\partial t_1(x)}{\partial x},$$

$$\sin t_1(x)\sin t_2(x) = \frac{4ar}{(a+r)^2}\sin x\sin\theta,$$

$$\frac{\sin t_1(x)}{\sin t_2(x)} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos\theta - \cos t_2(x)]} = \frac{\sin x\,[\cos t_1(x) - \cos\theta]}{\sin\theta\,[\cos x - \cos t_2(x)]}$$

$$\cos t_1(x)\cos t_2(x) = \frac{4ar\cos\theta\cos x - (r-a)^2}{(r+a)^2},$$

$$\cos t_1(x) + \cos t_2(x) = \frac{4ar}{(r+a)^2}(\cos x + \cos\theta),$$

where $t_1(x)$ and $t_2(x)$ are defined by (1.5.5)

CHAPTER 2

GENERALIZED POTENTIAL THEORY SOLUTIONS

We can generalize the Newton potential as $V = H/R^{1+\kappa}$, where R is the distance between two points, H is a constant depending on the physical constants of the space, and $-1 < \kappa < 1$. This potential has various applications in engineering, for example, in the theory of elasticity of inhomogeneous elastic body, with the modulus of elasticity E being a power function of z: $E = E_0 z^\kappa$. Other applications can be found in Weinstein (1952) and Payne (1952) where only the axisymmetric case was considered. Closed form solution to various non-axisymmetric problems is given in this Chapter.

2.1. Interior problem for a half-space

The problem is called interior when the potential is prescribed inside a circle.

Problem 1. We consider generalization of the problem solved in section 1.3. The boundary conditions are

$$V = f(\rho, \phi), \quad \text{for} \quad \rho \leq a, \quad 0 \leq \phi < 2\pi;$$

$$\sigma = 0, \quad \text{for}\, \rho > a, \quad 0 \leq \phi < 2\pi. \tag{2.1.1}$$

Here V is the generalized potential, and σ is the charge density distribution. The governing integral equation will take the form

$$H \int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0, \phi_0)\, \rho_0 d\rho_0 d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]^{(1+\kappa)/2}} = f(\rho, \phi), \tag{2.1.2}$$

Rostovtsev (1964) obtained an exact solution of (2) in Fourier series. Here we present a closed form solution.

By using the integral representation (1.2.1), integral equation (2) can be rewritten as

$$4H\cos\frac{\pi\kappa}{2}\int_0^\rho \frac{x^\kappa \mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)=f(\rho,\phi). \tag{2.1.3}$$

Integral equation (3) represents a sequence of two Abel operators and one $\mathcal{L}$-operator. The solution procedure is similar to that of (1.3.9). The first operator to be applied to both sides of (3) is

$$\mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho). \tag{2.1.4}$$

The result of application of (4) to both sides of (3) is

$$2\pi H t^\kappa\int_t^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-t^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{t}{\rho_0}\right)\sigma(\rho_0,\phi)=\mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho)f(\rho,\phi). \tag{2.1.5}$$

The second operator to be applied to both sides of (5) is

$$\mathcal{L}(y)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t^{1-\kappa}\mathrm{d}t}{(t^2-y^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{t}\right),$$

with the result

$$\sigma(y,\phi)=-\frac{\cos(\pi\kappa/2)}{\pi^2 H y}\mathcal{L}(y)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t^{1-\kappa}\mathrm{d}t}{(t^2-y^2)^{(1-\kappa)/2}}$$

$$\times\mathcal{L}\left(\frac{1}{t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho)f(\rho,\phi). \tag{2.1.6}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (6) in the form

$$\sigma(y,\phi)=\frac{\cos(\pi\kappa/2)}{\pi^2 H}\left[\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{(1-\kappa)/2}}-\int_y^a \frac{\mathrm{d}t}{(t^2-y^2)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{2.1.7}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t^{1+\kappa}} \int_0^t \frac{\rho^{1+\kappa}d\rho}{(t^2-\rho^2)^{(1-\kappa)/2}} \frac{d}{d\rho}\left[\rho^{1-\kappa}\mathcal{L}\left(\frac{\rho y}{t^2}\right)f(\rho,\phi)\right]. \qquad (2.1.8)$$

Yet another form of solution can be found in (Fabrikant, 1971e).

It becomes possible to compute various integral characteristics, like the total charge Q and the polarizability moments M_x and M_y directly in terms of the prescribed potential. Since

$$Q = \int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\,\rho d\rho d\phi, \qquad (2.1.9)$$

substitution of (6) in (9) yields directly the total charge

$$Q = \frac{\cos(\pi\kappa/2)}{\pi^2 H}\int_0^{2\pi}\int_0^a \frac{f(\rho,\phi)\,\rho d\rho d\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}. \qquad (2.1.10)$$

For computation of the moments M_x and M_y, it is convenient to introduce the complex parameter

$$M = M_x + iM_y = -i\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\,e^{i\phi}\,\rho^2 d\rho d\phi. \qquad (2.1.11)$$

By using (6), the final expression for the moment is found to be

$$M = -i\frac{2\cos(\pi\kappa/2)}{\pi^2 H(1+\kappa)}\int_0^{2\pi}\int_0^a \frac{f(\rho,\phi)\,e^{i\phi}\rho^2 d\rho d\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}. \qquad (2.1.12)$$

Expressions (11) and (12) are in agreement with similar results of Rostovtsev (1964).

By reviewing the derivation of expression (3), one may find that it is valid for evaluating the potential for $\rho > a$, if the upper limit of integration ρ is replaced by a. Substitution of (6) into the modified form of (3) results in

$$V(\rho,\phi)=\frac{2\cos(\pi\kappa/2)}{\pi}\int_0^a\frac{dx}{(\rho^2-x^2)^{(1+\kappa)/2}}\frac{d}{dx}\int_0^x\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)f(\rho_0,\phi),$$

$$\text{for } \rho>a. \tag{2.1.13}$$

Performing differentiation of the integrand, and then integrating by parts, we obtain

$$V(\rho,\phi)=\frac{1}{\pi^2}\cos(\frac{\pi\kappa}{2})(\rho^2-a^2)^{(1-\kappa)/2}\int_0^{2\pi}\int_0^a\frac{f(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{(1-\kappa)/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]},$$

$$\text{for } \rho>a. \tag{2.1.14}$$

Here the following identities were employed (Bateman and Erdélyi, 1955)

$$\frac{d}{d\zeta}\left[\zeta^{(1+\kappa)/2}F\left(\frac{1+\kappa}{2},\frac{1+\kappa}{2};\frac{3+\kappa}{2};\zeta\right)\right]=\frac{1+\kappa}{2}\zeta^{-(1-\kappa)/2}(1-\zeta)^{-(1+\kappa)/2},$$

$$\frac{d}{dx}\int_0^x\frac{f(t)t\,dt}{(x^2-t^2)^{(1-\kappa)/2}}=f(0)x^{\kappa}+x\int_0^x\frac{df(t)}{(x^2-t^2)^{(1-\kappa)/2}}. \tag{2.1.15}$$

All the quantities of interest, namely, the charge distribution σ, the total charge Q, the moment M, and the potential outside the circle can be expressed directly through the prescribed potential f by formulae (6), (10), (12), and (14) respectively.

Problem 2. In a similar manner, we can consider yet another mixed boundary value problem of a half-space, subject to the boundary conditions at $z=0$:

$$V=0,\quad \text{for}\quad 0\le\rho\le a,\ 0\le\phi<2\pi;$$

$$\sigma=\sigma(\rho,\phi),\quad \text{for } a<\rho<\infty,\ 0\le\phi<2\pi. \tag{2.1.16}$$

The governing integral equation takes the form (2), with the known function

$$f(\rho,\phi)=-H\int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}. \tag{2.1.17}$$

Its solution can be found in exactly the same manner as that of (1.4.27), and is

$$\sigma(\rho,\phi)=-\frac{\cos(\pi\kappa/2)}{\pi^2(a^2-\rho^2)^{(1-\kappa)/2}}\int_0^{2\pi}\int_a^{\infty}\frac{(\rho_0^2-a^2)^{(1-\kappa)/2}\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{2.1.18}$$

Expression (18) gives the charge distribution outside the circle through its prescribed value inside. Note that (1.4.30) may be considered as a particular case of (18), when $\kappa=0$.

The potential outside the circle can be evaluated as a superposition of the potential due to the charge inside the circle and the prescribed charge distribution outside. By using a procedure analogous to the one described in section 1.4, we can obtain the expression

$$V(\rho,\phi)=4H\cos\frac{\pi\kappa}{2}\left\{\int_\rho^{\infty}\frac{x^\kappa\mathrm{d}x}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_a^x\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)\right.$$

$$\left.+\int_0^a\frac{x^\kappa\mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^a\frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)\right\},\quad \text{for } \rho>a. \tag{2.1.19}$$

Substitution of (18) in (19) leads, after simplification, to

$$V(\rho,\phi)=4H\cos\frac{\pi\kappa}{2}\int_a^{\rho}\frac{x^\kappa\mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^{\infty}\frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi),\quad \text{for } \rho>a \tag{2.1.20}$$

The potential inside the circle is now defined in terms of the charge density distribution prescribed outside.

Introduce the *energy function*:

$$K_1(\rho,\phi)=\frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}}\int_\rho^{\infty}\frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-\rho^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma(\rho_0,\phi). \tag{2.1.21}$$

By using the property of the $\mathcal{L}$-operators (1.1.5) we may rewrite (20) as

$$V(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\int_a^\rho \frac{x^\kappa \mathrm{d}x}{(\rho^2 - x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)\int_x^\infty \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H\int_a^\rho \frac{x^{(1+\kappa)/2}\mathrm{d}x}{(\rho^2 - x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)K_1(x,\phi). \tag{2.1.22}$$

The energy W may be defined by the integral

$$W = \int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi)V(\rho,\phi)\rho\, \mathrm{d}\rho\mathrm{d}\phi. \tag{2.1.23}$$

Substitution of (22) in (23) gives

$$W = 2^{(3-\kappa)/2}\pi H\int_0^{2\pi}\mathrm{d}\phi\int_a^\infty \sigma(\rho,\phi)\rho\, \mathrm{d}\rho\int_a^\rho \frac{x^{(1+\kappa)/2}\,\mathrm{d}x}{(\rho^2 - x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H\int_0^{2\pi}\mathrm{d}\phi\int_a^\infty x^{(1+\kappa)/2}\mathrm{d}x\int_x^\infty \frac{\sigma(\rho,\phi)\rho\, \mathrm{d}\rho}{(\rho^2 - x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H\int_0^{2\pi}\mathrm{d}\phi\int_a^\infty K_1(x,\phi)x^{(1+\kappa)/2}\mathrm{d}x\int_x^\infty \frac{\rho\, \mathrm{d}\rho}{(\rho^2 - x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)\sigma(\rho,\phi). \tag{2.1.24}$$

Here the interchange of the order of integration was used twice. Now comparison of the last expression with (21) yields the final result

$$W = \frac{2^{1-\kappa}\pi^2 H}{\cos(\pi\kappa/2)}\int_0^{2\pi}\int_a^\infty [K_1(\rho,\phi)]^2\rho\, \mathrm{d}\rho\, \mathrm{d}\phi \tag{2.1.25}$$

Expression (25) gives a good explanation why we called K_1 energy function: its square is proportional to the energy per unit area. In the case of axial symmetry formula (25) simplifies as

$$W = \frac{2^{2-\kappa}\pi^3 H}{\cos(\pi\kappa/2)} \int_a^\infty [K_1(\rho)]^2 \rho\, d\rho,$$

with

$$K_1(\rho) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_\rho^\infty \frac{\sigma(\rho_0,\phi_0)\rho_0\, d\rho_0}{(\rho_0^2 - \rho^2)^{(1+\kappa)/2}}. \qquad (2.1.26)$$

2.2. Exterior problem for a half-space

We call a problem exterior when the potential is prescribed outside the circle $\rho = a$.

Problem 1. The problem is characterized by the following mixed boundary conditions on the plane z=0:

$$V = f(\rho,\phi), \quad \text{for} \quad \rho > a, \quad 0 \le \phi < 2\pi;$$

$$\sigma = 0, \quad \text{for} \quad \rho < a, \quad 0 \le \phi < 2\pi. \qquad (2.2.1)$$

Here the same notation is used as in the previous section. The governing integral equation will take the form

$$H \int_0^{2\pi} \int_a^\infty \frac{\sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}} = f(\rho,\phi), \qquad (2.2.2)$$

where $-1<\kappa<1$. The new method allows us to present a closed form solution.

By using the integral representation (1.2.17), for z=0, the integral equation (2) can be rewritten as

$$4H\cos\frac{\pi\kappa}{2} \int_\rho^\infty \frac{x^\kappa dx}{(x^2 - \rho^2)^{(1+\kappa)/2}} \int_a^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{(1+\kappa)/2}}\, \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi) = f(\rho,\phi). \qquad (2.2.3)$$

Integral equation (3) represents a sequence of two Abel operators and one $\mathcal{L}$-operator. The solution procedure is similar to that of section 1.4. The first operator to be applied to both sides of (3) is

$$\mathcal{L}(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{\rho}\right). \tag{2.2.4}$$

The result of application of (4) to both sides of (3) is

$$-2\pi H t^\kappa \int_a^t \frac{\rho_0\,\mathrm{d}\rho_0}{(t^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho_0}{t}\right)\sigma(\rho_0,\phi)$$

$$=\mathcal{L}(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{\rho}\right)f(\rho,\phi). \tag{2.2.5}$$

The second operator to be applied to both sides of (5) is

$$\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t^{1-\kappa}\mathrm{d}t}{(y^2-t^2)^{(1-\kappa)/2}}\mathcal{L}(t),$$

with the result

$$\sigma(y,\phi)=-\frac{\cos(\pi\kappa/2)}{\pi^2 H y}\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t^{1-\kappa}\mathrm{d}t}{(y^2-t^2)^{(1-\kappa)/2}}$$

$$\times\mathcal{L}(t^2)\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{\rho}\right)f(\rho,\phi). \tag{2.2.6}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (6) in the form

$$\sigma(y,\phi)=-\frac{\cos(\pi\kappa/2)}{\pi^2 H}\left[\frac{\Phi(a,y,\phi)}{(y^2-a^2)^{(1-\kappa)/2}}-\int_a^y \frac{\mathrm{d}t}{(y^2-t^2)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{2.2.7}$$

Here

$$\Phi(t,y,\phi) = t^{1-\kappa}\int_t^\infty \frac{d\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\frac{d}{d\rho}\left[\mathcal{L}\left(\frac{t^2}{\rho y}\right)f(\rho,\phi)\right]. \tag{2.2.8}$$

Certain integral characteristics can be computed directly in terms of the given function f. Since the total charge

$$Q = \int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi)\,\rho d\rho d\phi, \tag{2.2.9}$$

substitution of (6) in (9) yields directly the total charge

$$Q = \lim_{y\to\infty}\left\{-\frac{\cos(\pi\kappa/2)}{\pi^2 H}\int_a^y \frac{t^{2-\kappa}dt}{(y^2-t^2)^{(1-\kappa)/2}}\int_t^\infty \frac{d\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\frac{d}{d\rho}\int_0^{2\pi} f(\rho,\phi)\,d\phi\right\}. \tag{2.2.10}$$

The moment can be found in a similar manner. We can also express the potential inside the circle $\rho\le a$ directly in terms of the prescribed f outside the circle. We substitute (6) in (3) keeping in mind that, for $\rho\le a$, the lower limit of integration of the first integral will be a instead of ρ. By using the properties of Abel operators and the $\mathcal{L}$-operators, the following expression can be obtained

$$V(\rho,\phi) = -\frac{2}{\pi}\cos\frac{\pi\kappa}{2}\int_a^\infty \frac{dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\frac{d}{dx}\int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)f(\rho_0,\phi). \tag{2.2.11}$$

Carrying out the differentiation of the integrand, interchanging the order of integration, and then integrating with respect to x yields

$$V(\rho,\phi) = -\frac{2\cos(\pi\kappa/2)}{\pi(1+\kappa)}\int_a^\infty \left(\frac{\rho_0^2-a^2}{\rho_0^2-\rho^2}\right)^{(1+\kappa)/2}$$

$$\times F\left(\frac{1+\kappa}{2},\frac{1+\kappa}{2};\frac{3+\kappa}{2};\frac{\rho_0^2-a^2}{\rho_0^2-\rho^2}\right)\frac{d}{d\rho_0}\left[\mathcal{L}\left(\frac{\rho}{\rho_0}\right)f(\rho_0,\phi)\right]d\rho_0. \tag{2.2.12}$$

Integration by parts and the differential properties of the Gauss hypergeometric functions (2.1.15) allow us to simplify the last expression, namely,

$$V(\rho,\phi) = -\frac{2}{\pi}\cos\frac{\pi\kappa}{2}(a^2-\rho^2)^{(1-\kappa)/2}\int_a^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-a^2)^{(1-\kappa)/2}(\rho_0^2-\rho^2)}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)f(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)(a^2-\rho^2)^{(1-\kappa)/2}\int_0^{2\pi}\int_a^\infty \frac{f(\rho_0,\phi_0)\rho_0\, d\rho_0\, d\phi_0}{(\rho_0^2-a^2)^{(1-\kappa)/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]}. \quad (2.2.13)$$

Expression (13) gives the potential inside a circle $\rho\le a$ directly in terms of its prescribed value outside the circle. We note certain similarity between (2.1.14) and (13).

Problem 2. We consider yet another boundary value problem for a half-space, with the boundary conditions at $z=0$:

$$V = 0, \quad \text{for } a<\rho<\infty, \quad 0\le\phi<2\pi;$$

$$\sigma = \sigma(\rho,\phi), \quad \text{for } 0\le\rho\le a, \quad 0\le\phi<2\pi. \quad (2.2.14)$$

The governing integral equation takes the form (3), with the known function

$$f(\rho,\phi) = -H\int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}. \quad (2.2.15)$$

Its exact solution is

$$\sigma(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2(\rho^2-a^2)^{(1-\kappa)/2}}\int_0^{2\pi}\int_0^a \frac{(a^2-\rho_0^2)^{(1-\kappa)/2}\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \quad (2.2.16)$$

Again, one should notice the similarity between (2.1.18) and (16). Expression (16) gives the charge density distribution in the plane $z=0$ outside the circle in terms of the charge density given inside.

The potential can be evaluated as a superposition of the potentials due to the charge inside and outside of the circle. By using a procedure analogous to the one described in section 1.3, we obtain the expression

$$V(\rho,\phi,0) = 4H\cos\frac{\pi\kappa}{2}\left\{\int_a^\infty \frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_a^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)\right.$$

$$+\int_0^\rho \frac{x^\kappa dx}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)\Bigg\}, \text{ for } \rho<a. \qquad (2.2.17)$$

Substitution of (16) in (17) leads, after simplification, to

$$V(\rho,\phi)=4H\cos\frac{\pi\kappa}{2}\int_\rho^a \frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_0^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi), \qquad (2.2.18)$$

The potential is now found in terms of the given charge distribution.

We can again introduce the energy function

$$K_1(\rho,\phi)=\frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}}\int_0^\rho \frac{\rho_0 d\rho_0}{(\rho^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma(\rho_0,\phi). \qquad (2.2.19)$$

By using the property of the $\mathcal{L}$-operators (1.1.5) we may rewrite (18) as

$$V(\rho,\phi)=4H\cos\frac{\pi\kappa}{2}\int_\rho^a \frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho}{x}\right)\int_0^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma(\rho_0,\phi)$$

$$=2^{(3-\kappa)/2}\pi H\int_\rho^a \frac{x^{(1+\kappa)/2}dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho}{x}\right)K_1(x,\phi). \qquad (2.2.20)$$

The energy W may be defined by the integral

$$W=\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)V(\rho,\phi)\rho\, d\rho d\phi. \qquad (2.2.21)$$

Substitution of (20) in (21) gives

$$W=2^{(3-\kappa)/2}\pi H\int_0^{2\pi} d\phi\int_0^a \sigma(\rho,\phi)\rho\, d\rho\int_\rho^a \frac{x^{(1+\kappa)/2}dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho}{x}\right)K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} \mathrm{d}\phi \int_0^a x^{(1+\kappa)/2}\, \mathrm{d}x \int_0^x \frac{\sigma(\rho,\phi)\rho\, \mathrm{d}\rho}{(x^2-\rho^2)^{(1+\kappa)/2}}\, \mathcal{L}\!\left(\frac{\rho}{x}\right) K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} \mathrm{d}\phi \int_0^a K_1(x,\phi)\, x^{(1+\kappa)/2}\, \mathrm{d}x \int_0^x \frac{\rho\, \mathrm{d}\rho}{(x^2-\rho^2)^{(1+\kappa)/2}}\, \mathcal{L}\!\left(\frac{\rho}{x}\right) \sigma(\rho,\phi). \tag{2.2.22}$$

Here the interchange of the order of integration was used twice. Now comparison of the last expression with (19) yields the final result

$$W = \frac{2^{1-\kappa}\pi^2 H}{\cos(\pi\kappa/2)} \int_0^{2\pi} \int_0^a [K_1(\rho,\phi)]^2 \rho\, \mathrm{d}\rho\, \mathrm{d}\phi. \tag{2.2.23}$$

Expression (23) interprets the energy function squared as being proportional to the energy per unit area. In the case of axial symmetry, formula (23) simplifies to

$$W = \frac{2^{2-\kappa}\pi^3 H}{\cos(\pi\kappa/2)} \int_0^a [K_1(\rho)]^2 \rho\, \mathrm{d}\rho,$$

with

$$K_1(\rho) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_0^\rho \frac{\sigma(\rho_0,\phi_0)\rho_0\, \mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{(1+\kappa)/2}}. \tag{2.2.24}$$

2.3. Generalized problem for a spherical cap

We consider a spherical cap whose surface in spherical coordinates is given by $r = a$, $0 \le \theta \le \alpha$, $0 \le \phi < 2\pi$. Arbitrary potential $v(\theta,\phi)$ is prescribed at each point of the cap. The charge density $\sigma(\theta,\phi)$ is to be defined. The potential can be described through a simple layer distribution as

$$V(r,\theta,\phi) = \int_0^\alpha \int_0^{2\pi} \frac{\sigma(\theta_0,\phi_0)\, a^2 \sin\theta_0\, \mathrm{d}\theta_0 \mathrm{d}\phi_0}{\{r^2 + a^2 - 2ar[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)]\}^{(1+\kappa)/2}}. \tag{2.3.1}$$

Taken at the surface of the cap, expression (1) gives the governing integral equation

$$\frac{a^{1-\kappa}}{2^{(1+\kappa)/2}}\int_0^{\alpha}\int_0^{2\pi}\frac{\sigma(\theta_0,\phi_0)\sin\theta_0\, d\theta_0 d\phi_0}{\{1-[\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)]\}^{(1+\kappa)/2}} = v(\theta,\phi). \tag{2.3.2}$$

The denominator in (2) can be transformed as follows:

$$1-[\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)] = 2\cos^2\frac{\theta}{2}\cos^2\frac{\theta_0}{2}\left[\tan^2\frac{\theta}{2}+\tan^2\frac{\theta_0}{2}\right.$$

$$\left.-2\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\cos(\phi-\phi_0)\right]. \tag{2.3.3}$$

Substitution of (3) in (2) yields the governing equation in the form

$$\frac{a^{1-\kappa}}{2^{1+\kappa}}\int_0^{\alpha}\int_0^{2\pi}\frac{\sigma(\theta_0,\phi_0)\sin\theta_0\, d\theta_0 d\phi_0}{\left(\cos\frac{\theta}{2}\cos\frac{\theta_0}{2}\right)^{1+\kappa}\left[\tan^2\frac{\theta}{2}+\tan^2\frac{\theta_0}{2}-2\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\cos(\phi-\phi_0)\right]^{(1+\kappa)/2}} = v(\theta,\phi),$$

$$\text{for } 0\le\theta\le\alpha. \tag{2.3.4}$$

By using formally (1.2.1) we can write

$$\left[\tan^2\frac{\theta}{2}+\tan^2\frac{\theta_0}{2}-2\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\cos(\phi-\phi_0)\right]^{-(1+\kappa)/2}$$

$$=\frac{2}{\pi}\cos\frac{\pi\kappa}{2}\left(\cos\frac{\theta}{2}\cos\frac{\theta_0}{2}\right)^{1+\kappa}\int_0^{\min(\theta,\theta_0)}\frac{\lambda\left(\frac{\tan^2(\xi/2)}{\tan(\theta/2)\tan(\theta_0/2)},\phi-\phi_0\right)\sin^{\kappa}\xi\, d\xi}{[(\cos\xi-\cos\theta)(\cos\xi-\cos\theta_0)]^{(1+\kappa)/2}}. \tag{2.3.5}$$

Substitution of (5) in (4) yields

$$\frac{a^{1-\kappa}}{2^{\kappa-1}}\int_0^{\theta}\frac{\sin^{\kappa}\xi\, d\xi}{(\cos\xi-\cos\theta)^{(1+\kappa)/2}}\int_{\xi}^{\alpha}\frac{\sin\theta_0\, d\theta_0}{(\cos\xi-\cos\theta_0)^{(1+\kappa)/2}}\,\mathcal{L}\left(\frac{\tan^2(\xi/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi_0)$$

$$= v(\theta,\phi). \tag{2.3.6}$$

Again we have an integral equation consisting of two Abel-type operators and the $\mathcal{L}$-operator. We apply to both sides of (6) the following operator:

$$\mathcal{L}\left(\cot\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right). \tag{2.3.7}$$

The result is

$$2^{-\kappa}a^{1-\kappa}\sin^{\kappa}\theta_1\int_{\theta_1}^{\alpha}\frac{\sin\theta_0\,d\theta_0}{(\cos\theta_1-\cos\theta_0)^{(1+\kappa)/2}}\,\mathcal{L}\left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi_0)$$

$$=\mathcal{L}\left(\cot\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right)v(\theta,\phi). \tag{2.3.8}$$

The following well-known integral was used here

$$\int_{\theta_1}^{\theta}\frac{\sin\xi\,d\xi}{(\cos\xi-\cos\theta)^{(1+\kappa)/2}(\cos\theta_1-\cos\xi)^{(1-\kappa)/2}}=\frac{\pi}{\cos(\pi\kappa/2)}. \tag{2.3.9}$$

The next operator to apply is

$$\mathcal{L}\left(\tan\frac{\theta_2}{2}\right)\frac{d}{d\theta_2}\int_{\theta_2}^{\alpha}\frac{\sin^{1-\kappa}\theta_1\,d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\cot\frac{\theta_1}{2}\right),$$

with the result

$$\sigma(\theta_2,\phi)=-\frac{\cos(\pi\kappa/2)}{(2a)^{1-\kappa}\pi^2\sin\theta_2}\,\mathcal{L}\left(\tan\frac{\theta_2}{2}\right)\frac{d}{d\theta_2}\int_{\theta_2}^{\alpha}\frac{\sin^{1-\kappa}\theta_1\,d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\cot^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right)v(\theta,\phi). \tag{2.3.10}$$

Some further simplification of (10) is possible by using the following rules of differentiation

$$\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{f(\theta)\sin\theta\,d\theta}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}=\sin\theta_1\left[\frac{f(0)}{(1-\cos\theta_1)^{(1-\kappa)/2}}+\int_0^{\theta_1}\frac{df(\theta)}{d\theta}\frac{d\theta}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\right],$$

$$\frac{d}{d\theta_2}\int_{\theta_2}^{\alpha}\frac{f(\theta_1)\sin\theta_1\,d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}=\sin\theta_2\left[-\frac{f(\alpha)}{(\cos\theta_2-\cos\alpha)^{(1-\kappa)/2}}\right.$$

$$\left.+\int_{\theta_2}^{\alpha}\frac{df(\theta_1)}{d\theta_1}\frac{d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\right]. \tag{2.3.11}$$

The simplified result is

$$\sigma(\theta_2,\phi)=\frac{\cos(\pi\kappa/2)}{(2a)^{1-\kappa}\pi^2}\left\{\frac{\Phi(\alpha,\theta_2,\phi)}{(\cos\theta_2-\cos\alpha)^{(1-\kappa)/2}}-\int_{\theta_2}^{\alpha}\frac{d}{d\theta_1}[\Phi(\theta_1,\theta_2,\phi)]\frac{d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\right\},$$

with

$$\Phi(\theta_1,\theta_2,\phi)=\sin^{1-\kappa}\theta_1\left\{\frac{1}{(1-\cos\theta_1)^{(1-\kappa)/2}}\int_0^{2\pi}v(0,\phi)\,d\phi\right.$$

$$\left.+2\pi\int_0^{\theta_1}\frac{d\theta}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\frac{d}{d\theta}\left[\mathcal{L}\left(\frac{\tan(\theta_2/2)\tan(\theta/2)}{\tan^2(\theta_1/2)}\right)v(\theta,\phi)\right]\right\}. \tag{2.3.12}$$

In the case $\kappa\to 0$ a regular electrostatic solution is recovered.

It is of interest to establish a relationship between function g introduced by Collins (1959) and the axisymmetric component of σ. His function g was introduced by the expression

$$V(\theta)=\int_0^{\theta}\frac{g(\eta)\sec(\eta/2)\,d\eta}{\sqrt{2(\cos\eta-\cos\theta)}}. \tag{2.3.13}$$

Comparison of (13) with (6), taking into consideration σ does not depend on ϕ and $\kappa=0$, leads to the sought relationship

$$g(\eta) = 2\sqrt{2}\, a \cos\frac{\eta}{2} \int_{\eta}^{\alpha} \frac{\sigma(\theta_0) \sin\theta_0\, d\theta_0}{\sqrt{\cos\eta - \cos\theta_0}}. \tag{2.3.14}$$

2.4. Generalized potential problem for a surface of revolution

The following generalized potential problem is considered: given an arbitrary potential distribution on a surface of revolution, the charge density is to be determined. A closed form solution to the problem is obtained for a certain class of surfaces of revolution due to a special integral representation of the kernel of the governing integral equation. We consider an arbitrary surface of revolution, the axis $0z$ being the axis of revolution. Introduce a set of orthogonal curvilinear coordinates u and v, with their relations to the cartesian set as

$$x = f(u)\cos(v), \quad y = f(u)\sin(v), \quad z = w(u). \tag{2.4.1}$$

Here the functions f and w are known and are defined by the shape of the surface of revolution. The distance R between any two points (u,v) and (u_0,v_0) on the surface may be defined as

$$R^2 = [f(u)\cos(v) - f(u_0)\cos(v_0)]^2 + [f(u)\sin(v) - f(u_0)\sin(v_0)]^2$$
$$+ [w(u) - w(u_0)]^2 = h_1(u,u_0) - h(u,u_0)\cos(v - v_0);$$
$$h(u, u_0) = 2 f(u)f(u_0), \quad h_1(u, u_0) = f^2(u) + f^2(u_0) + [w(u) - w(u_0)]^2. \tag{2.4.2}$$

Let us represent

$$h_1(u,u_0) - h(u,u_0)\cos(v - v_0)$$
$$= h(u,u_0)\frac{[\xi_1(u,u_0) - 2\xi_1(u,u_0)\xi_2(u,u_0)\cos(v - v_0) + \xi_2(u,u_0)]}{2} \tag{2.4.3}$$

where

$$\xi_{1,2}(u,u_0) = \frac{h_1(u,u_0)}{h(u,u_0)} \pm \left[\frac{h_1^2(u,u_0)}{h^2(u,u_0)} - 1\right]^{1/2}. \tag{2.4.4}$$

Note that

$$\xi_1(u,u_0) = 1/\xi_2(u,u_0) \tag{2.4.5}$$

Using (2), (4) and (5), we establish also

$$\xi_1(u,u) = \xi_2(u,u) = 1 \tag{2.4.6}$$

Assume the existence of a function ζ such that

$$\xi_1(u,u_0) = \zeta(u)/\zeta(u_0), \quad \xi_2(u,u_0) = \zeta(u_0)/\zeta(u) \tag{2.4.7}$$

Later it will be shown how to find such a function for certain systems of coordinates. Substitution of (7) and (3) into (2) leads to the relationship

$$R^2 = \frac{f(u)\,f(u_0)}{\zeta(u)\,\zeta(u_0)}\,[\zeta^2(u) - 2\zeta(u)\zeta(u_0)\cos(v - v_0) + \zeta^2(u_0)]. \tag{2.4.8}$$

Now reformulating the problem as: given an arbitrary potential distribution $V(u,v)$ on the surface of revolution ($a \le u \le b$, $0 \le v < 2\pi$), the generalized charge distribution is to be determined. Using (8), one may write the governing integral equation in the form

$$\int_0^{2\pi}\int_a^b \frac{\sigma(u_0,v_0)g(u_0)du_0dv_0}{\}\,[f(u)f(u_0)/\zeta(u)\zeta(u_0)]\,[\zeta^2(u) - 2\zeta\,(u)\zeta(u_0)\cos(v - v_0) + \zeta^2(u_0)]\,\{^{(1+\kappa)/2}}$$

$$= V(u,v), \quad \text{for} \quad a \le u \le b, \quad 0 \le v < 2\pi \tag{2.4.9}$$

For the system of curvilinear coordinates (1), the Jacobian g is given by

$$g(u) = f(u)\,[(df(u)/du)^2 + (dw(u)/du)^2\,]^{1/2}. \tag{2.4.10}$$

Making use of the integral representation established in (1.2.1)

$$\frac{1}{[r^2 - 2rr_0\cos(v - v_0) + r_0^2]^{(1+\kappa)/2}}$$

$$= \frac{2}{\pi}\cos\left(\frac{\pi\kappa}{2}\right)\int_0^{min(r,r_0)} \frac{\lambda\left(\dfrac{x^2}{rr_0}, v - v_0\right)x^\kappa dx}{[(r^2 - x^2)(r_0^2 - x^2)]^{(1+\kappa)/2}}. \tag{2.4.11}$$

Assuming $r \equiv \zeta(u)$, $r_0 \equiv \zeta(u_0)$ and $x \equiv \zeta(t)$, one will have instead of (11),

$$\frac{1}{[\zeta^2(u)-2\zeta(u)\zeta(u_0)\cos(v-v_0)+\zeta^2(u_0)]^{(1+\kappa)/2}}$$

$$=\frac{2}{\pi}\cos(\frac{\pi\kappa}{2})\int_c^{min(u,u_0)}\frac{\chi(t,u,u_0,v,v_0)\,\zeta^\kappa(t)\,d\zeta(t)}{\}[\zeta^2(u)-\zeta^2(t)][\zeta^2(u_0)-\zeta^2(t)]\{^{(1+\kappa)/2}} \tag{2.4.12}$$

Here, $c=\zeta^{-1}(0)$,

$$\chi(t,u,u_0,v,v_0)=\lambda[\zeta^2(t)/\zeta(u)\zeta(u_0),v-v_0], \tag{2.4.13}$$

and ζ^{-1} denotes the function inverse of ζ. Also, let

$$\zeta^{-1}(0)=a. \tag{2.4.14}$$

Later it will be shown that this requirement is not so limiting. Now substitution of (14) and (12) into (9) leads to the governing integral equation

$$\frac{2}{\pi}\cos(\frac{\pi\kappa}{2})\int_a^u\frac{\zeta^\kappa(t)d\zeta(t)}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}\int_t^b\frac{G(u_0)du_0}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}$$

$$\times\int_0^{2\pi}\chi(t,u,u_0,v,v_0)\,\sigma(u_0,v_0)dv_0=\Omega(u,v). \tag{2.4.15}$$

Here

$$G(u_0)=[\zeta(u_0)/f(u_0)]^{(1+\kappa)/2}g(u_0),\quad \Omega(u,v)=[f(u)/\zeta(u)]^{(1+\kappa)/2)}V(u,v). \tag{2.4.16}$$

The following scheme of change of the order of integration was employed:

$$\int_a^b du_0\int_a^{min(u,u_0)}dt=\int_a^u du_0\int_a^{u_0}dt+\int_u^b du_0\int_a^u dt$$

$$=\int_a^u dt\int_t^u du_0+\int_a^u dt\int_u^b du_0=\int_a^u dt\int_t^b du_0.$$

Equation (15), despite looking more complicated than the original equation (9), admits exact solution in a closed form, using a technique developed in previous sections. Equation (15) may be rewritten by introducing the $\mathcal{L}$-operators

$$4\cos(\frac{\pi\kappa}{2})\int_a^u \frac{\zeta^{\kappa}(t)d\zeta(t)}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}\int_t^b \frac{G(u_0)du_0}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}\times$$

$$\times\mathcal{L}\left(\frac{\zeta^2(t)}{\zeta(u)\zeta(u_0)}\right)\sigma(u_0,v)=\Omega(u,v)\quad \text{for}\quad a\le u\le b,\quad 0\le v\le 2\pi. \tag{2.4.17}$$

Now the governing equation (17) presents a sequence of two integral operators of Abel type and the $\mathcal{L}$-operator. The inverse operator to each one is known. An exact solution of (17) may now be constructed in several steps. Convergency at each step is not verified, but is assumed. Application of the operator

$$\mathcal{L}\left(\frac{1}{\zeta(u_1)}\right)\frac{d}{du_1}\int_a^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\mathcal{L}(\zeta(u))$$

to both sides of (17) gives

$$2\pi\zeta'(u_1)\,\zeta^{\nu}(u_1)\int_{u_1}^b \frac{G(u_0)du_0}{[\zeta^2(u_0)-\zeta^2(u_1)]^{(1-\kappa)/2}}\mathcal{L}\left(\frac{\zeta(u_1)}{\zeta(u_0)}\right)\sigma(u_0,v)$$

$$=\mathcal{L}\left(\frac{1}{\zeta(u_1)}\right)\frac{d}{du_1}\int_a^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\mathcal{L}[\zeta(u)]\,\Omega(u,v). \tag{2.4.18}$$

Hereafter, a prime indicates the derivative with respect to the variable given in brackets. The following integral is also used

$$\int_t^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\nu)/2}[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}=\frac{\pi}{2\cos(\pi\kappa/2)}. \tag{2.4.19}$$

The next step is application of the operator

$$\mathcal{L}[\zeta(u_2)]\frac{d}{du_2}\int_{u_2}^b\frac{\zeta^{1-\kappa}(u_1)du_1}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{\zeta(u_1)}\right)$$

to both sides of equation (18). The result then is

$$\sigma(u_2,v) = -\frac{\cos(\pi\kappa/2)}{\pi^2 G(u_2)}\,\mathcal{L}(\zeta(u_2))\,\frac{d}{du_2}\int_{u_2}^{b}\frac{\zeta^{1-\kappa}(u_1)du_1}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}$$

$$\times\,\mathcal{L}\left(\frac{1}{\zeta^2(u_1)}\right)\frac{d}{du_1}\int_a^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\,\mathcal{L}(\zeta(u))\,\Omega(u,v). \qquad (2.4.20)$$

One may establish the following rules of differentiation under the integral sign:

$$\frac{d}{du_1}\int_a^{u_1}\frac{\phi(u)\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}} = \left\{\frac{\phi(a)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\right.$$

$$\left. + \int_a^{u_1}\frac{d\phi(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\right\}\zeta(u_1)\,\zeta'(u_1),$$

$$\frac{d}{du_2}\int_{u_2}^{b}\frac{\phi(u_1)\zeta(u_1)d\zeta(u_1)}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}} = \left\{-\frac{\phi(b)}{[\zeta^2(\mathrm{b})-\zeta^2(u_2)]^{(1-\kappa)/2}}\right.$$

$$\left. + \int_{u_2}^{b}\frac{d\phi(u_1)}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}\right\}\zeta(u_2)\,\zeta'(u_2). \qquad (2.4.21)$$

Further simplification of (20) is possible by using (21). The result then is

$$\sigma(u_2,v) = \frac{\cos(\pi\kappa/2)}{2\pi^3\,G\,(u_2)}\,\zeta(u_2)\,\zeta'\,(u_2)\left\{\frac{\Phi(b,u_2,v)}{[\zeta^2(b)-\zeta^2(u_2)]^{(1-\kappa)/2}}\right.$$

$$\left. - \int_{u_2}^{b}\frac{[d\Phi(u_1,u_2,v)\,/du_1]}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}\,du_1\right\},$$

$$\Phi(u_1, u_2, v) = \int_0^{2\pi}]\Omega(a,v)\mathrm{d}v$$

$$+ 2\pi\zeta^{1-\kappa}(u_1)\int_a^{u_1} \frac{du}{[\zeta^2(u_1) - \zeta^2(u)]^{(1-\kappa)/2}} \frac{d}{du}\left[\mathcal{L}\left(\frac{\zeta(u)\zeta(u_2)}{\zeta^2(u_1)} \right)\Omega(u,v) \right]. \qquad (2.4.22)$$

The solution of the problem is now complete; expressions (20) and (22) give the two different forms of presentation of the solution.

The method of exact solution to the generalized potential problem, proposed in this section, may be applied only to those systems of curvilinear coordinates, which satisfy the requirements (7) and (14). This limitation is not that severe, as it might appear. We may consider a spherical cap as an example of a surface of revolution. If the cap radius is r, then

$$f(u) = r\sin u, \quad w(u) = r\cos u, \quad a = 0,$$

and formulae (2), (4) and (7) give

$$\xi_1(u,u_0) = \frac{\tan(u/2)}{\tan(u_0/2)} = \frac{1}{\xi_2(u,u_0)}, \quad \zeta(u) = \tan\frac{u}{2}.$$

Since $\zeta^{-1}(u) = 2\tan^{-1}u$, the requirement in (14) is satisfied and the solution (22) is valid.

This does not mean that the system of spherical coordinates is the only one to satisfy conditions (7) and (14). For example, the system of polar coordinates on a circular disk also satisfies (7) and (14); namely, for that case

$$f(u) = u, \quad w(u) = 0, \quad a = 0, \quad \xi_1(u,u_0) = 1/\xi_2(u,u_0) = u/u_0, \quad \zeta(u) = u$$

and formula (22) gives the solution of the generalized potential problem for a circular disk, corresponding to the one obtained in previous section.

When the surface of revolution is such that the system of curvilinear coordinates does not satisfy the requirements (7) and (14), the procedure of approximation of (4) in the form (7) may be recommended. If the procedure is successful, formula (22) will give an approximate solution to the problem. As an example of such an approximation, consider the case of a cylinder of radius R and height H. In this case $f(u) = R$, $w(u) = u$ and formulae (2) and (4) give

$$\xi_{1,2}(u,u_0) = 1 + \frac{1}{2}\left(\frac{u-u_0}{R}\right)^2 \pm \left(\frac{u-u_0}{R}\right)\left[1 + \left(\frac{u-u_0}{2R}\right)^2\right]^{1/2}. \qquad (2.4.23)$$

If the condition $(H/R) \ll 1$ holds, the following approximation of (23) is valid:

$$\xi_{1,2}(u,u_0) = 1 \pm \left(\frac{u-u_0}{R}\right)$$

The last expression may be represented with the same degree of accuracy as

$$\xi_1(u,u_0) = \frac{R+u}{R+u_0}, \quad \xi_2(u,u_0) = \frac{R+u_0}{R+u}.$$

Now one may assume $\zeta(u)=R+u$, $a=-R$ and the requirements (7) and (14) are satisfied, then all the results of this section become applicable to the case of a short cylinder.

It may also be noticed that it is possible to solve the mathematically identical problems in hydrodynamics, nonhomogeneous elasticity, heat conduction, etc., using the same general approach.

Exercises 2

1. The potential inside a circle is given by $V(\rho,\phi)=w_0+\theta\rho\cos\phi$, with w_0=const, and θ=const. Find the corresponding charge distribution σ.
Answer:

$$\sigma(\rho,\phi) = \frac{\cos(\pi\kappa/2)}{\pi^2 H}\,\frac{w_0 + (2\theta\rho\cos\phi)/(1+\kappa)}{(a^2-\rho^2)^{(1-\kappa)/2}}.$$

2. In the problem above express the total charge Q and the moment M, in terms of the parameters w_0 and θ.
Answer:

$$Q = \frac{2w_0 a^{1+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)}, \quad M = \frac{4\theta a^{3+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)(3+\kappa)}.$$

3. The potential inside a circle is given by $V(\rho,\phi)=w_0-c\rho^2$, with w_0=const, and c=const. Find the charge distribution σ and the radius a, if it is known that $\sigma(a,\phi)=0$.
Solution: utilization of (2.1.6) yields

$$\sigma(\rho)=\frac{\cos(\pi\kappa/2)}{\pi^2 H(a^2-\rho^2)^{(1-\kappa)/2}}\left\{w_0+\frac{2c[a^2(1-\kappa)-2\rho^2]}{(1+\kappa)^2}\right\}.$$

The radius a is found from the condition $\sigma(a)=0$. The result is $a=[(1+\kappa)w_0/(2c)]^{1/2}$, and the final expression for the charge distribution is

$$\sigma(\rho)=\frac{2w_0\cos(\pi\kappa/2)}{\pi^2 a^2 H(1+\kappa)}(a^2-\rho^2)^{(1+\kappa)/2}.$$

4. In the problem above find the total charge Q.
Answer:

$$Q=\frac{4w_0\,a^{1+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)(3+\kappa)}.$$

5. A circular disk of radius a is grounded and kept at zero potential. A point charge P is present at (b,ψ), $b>a$. Find the induced charge distribution.
Answer:

$$\sigma(\rho,\phi)=-\frac{P}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)\left(\frac{b^2-a^2}{a^2-\rho^2}\right)^{(1-\kappa)/2}\frac{1}{\rho^2+b^2-2b\rho\cos(\phi-\psi)}.$$

6. A circular disk of radius a is grounded and kept at zero potential. Let a uniform density charge σ_0 be prescribed at the annulus $b\le\rho\le c$, $(b>a)$, with no active charge elsewhere. Find the charge density σ for $\rho<a$.
Answer:

$$\sigma(\rho)=-\frac{2\sigma_0\cos(\pi\kappa/2)}{\pi(a^2-\rho^2)^{(1-\kappa)/2}}\int_b^c\frac{(\rho_0^2-a^2)^{(1-\kappa)/2}\rho_0\,d\rho_0}{\rho_0^2-\rho^2},\quad \text{for } \rho<a.$$

In general, the last integral can be computed in terms of hypergeometric functions. In the particular case of $\kappa=0$, the integral is computable in elementary functions:

$$\sigma(\rho)=-\frac{2}{\pi}\sigma_0\left[\frac{(c^2-a^2)^{1/2}-(b^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}}-\tan^{-1}\left(\frac{c^2-a^2}{a^2-\rho^2}\right)^{1/2}\right.$$

$$\left.+\tan^{-1}\left(\frac{b^2-a^2}{a^2-\rho^2}\right)^{1/2}\right].$$

7. In the preceding problem find the potential V.
Answer:

$$V(\rho)=\frac{4\cos(\pi\kappa/2)}{1-\kappa}\sigma_0\left\{\int_a^{\min(\rho,c)}\frac{(c^2-x^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1+\kappa)/2}}x^\kappa\,\mathrm{d}x\right.$$

$$\left.-\int_a^{\min(\rho,b)}\frac{(b^2-x^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1+\kappa)/2}}x^\kappa\,\mathrm{d}x\right\},\text{ for }\rho>a.$$

8. By using formula (2.1.6), prove the identity

$$\int_0^{2\pi}\int_0^a\sigma(\rho,\phi)\rho^{1+|n|}\,\mathrm{e}^{in\phi}\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$=\frac{(1+\kappa)\Gamma(1+|n|)}{2\pi^2\Gamma[|n|+(1+\kappa)/2]}\cos\frac{\pi\kappa}{2}\int_0^{2\pi}\int_0^a\frac{f(\rho,\phi)\rho^{1+|n|}\,\mathrm{e}^{in\phi}\,\mathrm{d}\rho\,\mathrm{d}\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}.$$

Note: in the particular cases $n=0$ and $n=-1$, the last identity transforms into (2.1.10) and (2.1.12) respectively.

9. Define the energy function $K_1(\rho,\phi)$ in terms of the potential V for interior problems.
Answer:

$$K_1(\rho,\phi)=\frac{\cos(\pi\kappa/2)}{2^{(1-\kappa)/2}\pi^2H\rho^{(1+\kappa)/2}}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho\frac{x\,\mathrm{d}x}{(\rho^2-x^2)^{(1-\kappa)/2}}\mathcal{L}(x)V(x,\phi).$$

Hint: perform the inversion of (2.1.22).

10. Prove the identity

$$\lim_{\rho\to a}\left\{\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho\frac{(x^2-a^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1-\kappa)/2}}f(x)\,\mathrm{d}x\right\}=\frac{\pi(1-\kappa)f(a)}{2\cos(\pi\kappa/2)}.$$

Hint: use the substitution $t=(\rho^2-x^2)/(x^2-a^2)$.

11. The following charge density distribution is prescribed over a circle $\rho\le a$: $\sigma(\rho,\phi)=\sigma_0+\sigma_1\rho\cos\phi$, with $\sigma_0=$const and $\sigma_1=$const. The diaphragm outside the circle is grounded. Find the charge induced on the diaphragm.
Answer:

$$\sigma(\rho,\phi) = -\frac{2\cos(\pi\kappa/2)}{\pi(3-\kappa)}\left(\frac{a}{\rho}\right)^{3-\kappa}\left\{\sigma_0 F\left(\frac{3-\kappa}{2},\frac{3-\kappa}{2};\frac{5-\kappa}{2};(\frac{a}{\rho})^2\right)\right.$$

$$\left. +\frac{2}{5-\kappa}\left(\frac{a}{\rho}\right)^2 F\left(\frac{3-\kappa}{2},\frac{5-\kappa}{2};\frac{7-\kappa}{2};(\frac{a}{\rho})^2\right)\sigma_1\rho\cos\phi\right\}, \text{ for } \rho>a.$$

The result can be expressed in elementary functions for $\kappa=0$

$$\sigma(\rho,\phi) = -\frac{2}{\pi}\left\{\left[\frac{a}{(\rho^2-a^2)^{1/2}} - \sin^{-1}(\frac{a}{\rho})\right]\sigma_0\right.$$

$$\left. +\left[\frac{3\rho^2-a^2}{3\rho(\rho^2-a^2)^{1/2}} - 3\frac{\rho}{a}\sin^{-1}(\frac{a}{\rho})\right]\sigma_1\rho\cos\phi\right\}.$$

12. In the problem above find the potential V.
Answer:

$$V = \frac{4\cos(\pi\kappa/2)}{(1-\kappa)^2}(a^2-\rho^2)^{(1-\kappa)/2}\left[\sigma_0 + \frac{2}{3-\kappa}\sigma_1\rho\cos\phi\right].$$

13. A point charge P is located at the point (b,ψ), $b<a$. The diaphragm outside the circle is grounded. Find the charge induced on the diaphragm.
Answer:

$$\sigma(\rho,\phi) = -\frac{P}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)\left(\frac{a^2-b^2}{\rho^2-a^2}\right)^{(1-\kappa)/2}\frac{1}{\rho^2+b^2-2b\rho\cos(\phi-\psi)}.$$

14. Express the energy function $K_1(\rho,\phi)$ in terms of the potential V in exterior problems.
Answer:

$$K_1(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{2^{(1-\kappa)/2}\pi^2 H\rho^{(1+\kappa)/2}}\mathcal{L}(\rho)\frac{d}{d\rho}\int_\rho^a \frac{x\,dx}{(x^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{x}\right)V(x,\phi).$$

15. In the generalized potential theory, consider a spherical cap of radius r with uniform potential prescribed over the surface, namely $V(u,v)=V_0=\text{const.}$ Find the charge density distribution.
Answer:

$$\sigma(u_2, v) = \frac{V_0\cos(\pi\kappa/2)}{(2r)^{1-\kappa}\pi^2}\left\{\frac{1-\kappa}{1+\kappa}z^{(1+\kappa)/2}F\left(\frac{1+\kappa}{2},\frac{1+\kappa}{2},\frac{3+\kappa}{2};z\right)\right.$$

$$\left.+\frac{\cos^{1-\kappa}(b/2)}{[\cos^2(u_2/2)-\cos^2(b/2)]^{(1-\kappa)/2}}\right\},$$

$$z = 1-\frac{\cos^2(b/2)}{\cos^2(u_2/2)}.$$

Here $F(\alpha,\ \beta;\ \gamma;\ z)$ is the Gauss hypergeometric function, and the following integral was employed.

$$\int_{u_2}^{b}\frac{\cos^{\beta}\left(\frac{u_1}{2}\right)d(\cos(u_1/2))}{\left[\cos^2\left(\frac{u_2}{2}\right)-\cos^2\left(\frac{u_1}{2}\right)\right]^{(1-\kappa)/2}} = -\frac{\cos^{\beta-1}(u_2/2)}{1+\kappa}\left[\cos^2\left(\frac{u_2}{2}\right)-\cos^2\left(\frac{b}{2}\right)\right]^{(1+\kappa)/2}$$

$$\times F\left(\frac{1-\beta}{2},\frac{1+\kappa}{2};\frac{3+\kappa}{2};\ z\right).$$

16. In the generalized potential theory consider the case of a grounded spherical cap or radius r in a uniform electric field of intensity E, acting along the $0z$ axis. For this case, $V(u,v) = Er\cos u$. Find the charge density distribution.

Answer:

$$\sigma(u_2,v) = \frac{Er^{\kappa}\cos(\pi\kappa/2)}{2^{1-\kappa}(1+\kappa)\pi^2}\left\{\left(\cos^{1-\kappa}\frac{b}{2}\right)\frac{(3-\kappa)\cos u_2-(1-\kappa)(1+\cos b)}{(\cos^2(u_2/2)-\cos^2(b/2))^{(1-\kappa)/2}}\right.$$

$$\left.+\frac{(3-\kappa)(1-\kappa)}{1+\kappa}z^{(1+\kappa)/2}\cos u_2\,F\left(\frac{1+\kappa}{2},\frac{1+\kappa}{2};\frac{3+\kappa}{2};z\right)\right\},$$

$$z = 1-\frac{\cos^2(b/2)}{\cos^2(u_2/2)}$$

Here integral from Example 15 was employed, as well as the following properties of the hypergeometric functions

$$F\left(-\frac{1-\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)$$

$$= \frac{1+\kappa}{2}(1-z)^{(1-\kappa)/2} + \frac{1-\kappa}{2} F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right),$$

$$F\left(-\frac{3-\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)$$

$$= \frac{(1-\kappa)(3-\kappa)}{8} F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)$$

$$-\frac{1+\kappa}{8}(2z-5+\kappa)(1-z)^{(1-\kappa)/2},$$

$$F(d, \beta; \beta; z) = (1-z)^{-d}.$$

17. In the generalized potential theory, consider the problem of a grounded spherical cap in a uniform electrical field, acting in the $0x$ direction. For this case $V(u,v) = Er\sin u\cos v$. Find the charge density distribution

Answer:

$$\sigma(u_2, v) = \frac{2E(2r)^{\kappa}\cos(\pi\kappa/2)}{\pi^2(1+\kappa)} \cos v \tan\left(\frac{u_2}{2}\right)$$

$$\times\left\{\frac{3-\kappa}{1+\kappa}\cos^2\left(\frac{u_2}{2}\right) z^{(1+\kappa)/2} F\left(-\frac{1-\kappa}{2}, \frac{1+\kappa}{2}, \frac{3+\kappa}{2}; z\right)\right.$$

$$\left. + \frac{\cos^{3-\kappa}(b/2)}{[\cos^2(u_2/2) - \cos^2(b/2)]^{(1-\kappa)/2}}\right\},$$

with

$$z = 1 - \frac{\cos^2(b/2)}{\cos^2(u_2/2)}$$

CHAPTER 3

APPLICATIONS IN ELECTROMAGNETICS

The general results of Chapter 1 are applied here to investigation of interaction of several charged coaxial and arbitrarily located disks. New type of governing integral equation is derived for the Dirichlet and Neumann problems for a circular annulus domain. Simple yet accurate formulae are derived for the capacity of flat laminae. Similar results were obtained for the electrical and magnetic polarizability of small apertures of general shape.

3.1. Interaction of several coaxial disks

The problem of charged coaxial circular disks has been attracting the attention of scientists during the last century. Kirchhoff, Ignatowsky, Love, Nomura, Cooke and others made a significant contribution to its solution. A comprehensive literature review can be found in (Ufliand, 1977) and (Leppington and Levine, 1970). The latest published important result seems to be (Kuz'min, 1971) where an axisymmetric problem of several charged coaxial disks is considered by the dual integral equation method. The solution is obtained for the case when the distance between disks is long in comparison with their radii.

A general solution to the non-axisymmetric problem is given here by a new approach. A mathematical outline of the method is given first. Formulation of the problem and its solution follows. The charge density distribution can be found from a system of Fredholm integral equation. The total charge and some other integral characteristics can be estimated without solving the system. The collocation method is used for the numerical solution of the integral equation. A way is found to assess the error of the computation. A high degree of accuracy of the solution allowed us to correct some numerical results published by Cooke (1958). Several examples are considered.

Preliminaries. Some mathematical considerations are presented here to simplify understanding of the method. Introduce the following quantities:

$$l_{1,2}(x,\rho,z) \equiv l_{1,2}(x) = \tfrac{1}{2}[\sqrt{(x+\rho)^2+z^2} \mp \sqrt{(x-\rho)^2+z^2}\,], \qquad (3.1.1)$$

$$l_{10,20}(x) \equiv l_{1,2}(x,\rho_0,z_0). \tag{3.1.2}$$

The following integral is valid

$$\int T(x)\,T^0(x)\,\lambda\left(\frac{l_1(x)\,l_{10}(x)}{l_2(x)\,l_{20}(x)},\,\phi-\phi_0\right)\mathrm{d}x = -\frac{1}{2R_1}\tan^{-1}\frac{\eta_1(x)}{R_1} - \frac{1}{2R_2}\tan^{-1}\frac{\eta_2}{R_2}. \tag{3.1.3}$$

Here

$$T(x) = \frac{1}{\sqrt{\rho^2 - l_1^2(x)}}\frac{\partial l_1(x)}{\partial x} = \frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}, \tag{3.1.4}$$

$$T^0(x) = \frac{1}{(\rho_0^2-(l_{10}^2(x))^{1/2}}\frac{\partial l_{10}}{\partial x} = \frac{\sqrt{l_{20}^2(x)-x^2}}{l_{20}^2(x)-l_{10}^2(x)}, \tag{3.1.5}$$

$$\lambda(k,\phi-\phi_0) = \frac{1-k^2}{1+k^2-2k\cos(\phi-\phi_0)}, \tag{3.1.6}$$

$$R_{1,2}^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z\mp z_0)^2, \tag{3.1.7}$$

$$\eta_{1,2}(x) = S(x) \pm zz_0/S(x), \tag{3.1.8}$$

$$S(x) = (l_2^2(x)-x^2)^{1/2}(l_{20}^2(x)-x^2)^{1/2}/x. \tag{3.1.9}$$

Formula (3) can be verified using the following relationships

$$\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)},\,\phi-\phi_0\right)$$

$$= \frac{1}{2x^2}\left[\,l_2^2(x)\,l_{20}^2(x)-l_1^2(x)\,l_{10}^2(x)\right]\left[\frac{1}{R_1^2+\eta_1^2(x)}+\frac{1}{R_2^2+\eta_2^2(x)}\right], \tag{3.1.10}$$

$$\frac{\partial}{\partial x}S(x) = -\frac{S(x)[l_2^2(x)\,l_{20}^2(x)-l_1^2(x)l_{10}^2(x)]}{x[l_2^2(x)-l_1^2(x)][(l_{20}^2(x)-(l_{10}^2(x)]}, \tag{3.1.11}$$

$$l_2(x)l_2(x) = x\rho, \qquad l_1^2(x)+l_2^2(x) = x^2+\rho^2+z^2. \tag{3.1.12}$$

Several particular cases of (3) will also be used in this section. For example, setting $z = 0$ in (3) one gets

$$\int \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} T^0(x)\, \lambda\left(\frac{l_{10}^2(x)}{\rho\rho_0}, \phi - \phi_0\right) = -\frac{1}{R} \tan^{-1} \frac{\sqrt{\rho^2 - x^2}\, [l_{20}^2(x) - x^2]^{1/2}}{xR}. \tag{3.1.13}$$

where

$$R^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z_0^2.$$

Another variation of (13) can be obtained by a substitution $y = l_1(x)$. The result is

$$\int \frac{\mathrm{d}y}{(\rho^2 + g^2(y))^{1/2}(\rho_0^2 - y^2)^{1/2}} \lambda\left(\frac{y^2}{\rho\rho_0}, \phi - \phi_0\right) = -\frac{1}{R} \tan^{-1} \frac{(\rho^2 + g^2(y))^{1/2}(\rho_0^2 - y^2)^{1/2}}{yR}. \tag{3.1.14}$$

Here

$$g(y) = y\sqrt{1 + z_0^2/(\rho_0^2 - y^2)}. \tag{3.1.15}$$

Formulation of the problem. We consider a system of n charged circular coaxial disks. We place a set of the cylindrical coordinate axes so that the axis Oz passes through the centers of the disks. Let a_i be the radius of the i-th disk, z_i be the z-coordinate of its center. The problem is to find the electrostatic field potential of the system of charged disks, i.e. to find a harmonic function $V(\rho,\phi,z)$ subject to the following boundary conditions:

$$V(\rho,\phi,z_k) = v_k(\rho,\phi) \quad \text{for } 0 \le \rho \le a_k, \quad 0 \le \phi < 2\pi, \quad k = 1,2,...,n. \tag{3.1.16}$$

The potential can be represented through the simple layer distribution as follows:

$$V(\rho,\phi,z) = \sum_{i=1}^{n} \int_0^{2\pi}\int_0^{a_i} \frac{q_i(\rho_0,\phi_0)\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + (z - z_i)^2}}. \tag{3.1.17}$$

Here q_i are yet unknown charge densities which can be found from a system of integral equations obtained by substitution of the boundary conditions (16) in (17). Now some transformations of (17) are necessary. Introduce the notations

$$l_{ik,1}(x,\rho_0,h_{ik}) \equiv l_{ik,1}(x) = l_1(x,\rho_0,h_{ik}), \tag{3.1.18}$$

$$h_{ik} = |z_i - z_k|. \tag{3.1.19}$$

Expressions (13) and (14) allows the following integral representations for the reciprocal distance

$$\frac{1}{R_{ik}} = \frac{2}{\pi}\int_0^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} T_{ik}(x,\rho_0)\lambda\left(\frac{l_{ik,1}^2(x)}{\rho\rho_0}, \phi - \phi_0\right), \tag{3.1.20}$$

$$\frac{1}{R_{ik}} = \frac{2}{\pi}\int_0^{l_{ik,1}(\rho)} \frac{dx}{(\rho^2 - g_{ik}^2(x))^{1/2}\sqrt{\rho_0^2 - x^2}}\, \lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0), \tag{3.1.21}$$

where $g_{ik}(x)$ according to (15) can be defined as

$$g_{ik}(x) = x\sqrt{1 + h_{ik}^2/(\rho_0^2 - x^2)}, \tag{3.1.22}$$

and

$$R_{ik}^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + h_{ik}^2, \tag{3.1.23}$$

$$T_{ik}(x,\rho_0) = \frac{1}{\sqrt{\rho_0^2 - l_{ik,1}^2(x)}}\frac{\partial}{\partial x} l_{ik,1}(x). \tag{3.1.24}$$

Now we can single out the k-th disk, without loss of generality, and do all the transformations of the integral equation related to the boundary conditions at the surface of the k-th disk, having in mind that the integral equations related to the other disks can be transformed in a similar manner. Substituting the boundary condition (16) in (17) and using (20) and (21), one obtains

$$4\int_0^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^{a_k} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) q_k(\rho_0,\phi)$$
$$+ \sum_{i=1, i\neq k}^{n} 4\int_0^{a_i} \rho_0 d\rho_0 \left\{\int_0^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} T_{ik}(x,\rho_0)\mathcal{L}\left(\frac{l_{ik,1}^2(x)}{\rho\rho_0}\right)\right\} q_i(\rho_0,\phi) = v_k(\rho,\phi).$$
(3.1.25)

Equation (25) can be transformed to the Fredholm type by the following procedure. Apply the operator

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_0^{r} \frac{\rho d\rho}{\sqrt{r^2 - \rho^2}} \mathcal{L}(\rho)$$

to both sides of (25). The result is

$$2\pi\int_r^{a_k}\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-r^2}}\mathcal{L}\left(\frac{r}{\rho_0}\right)q_k(\rho_0,\phi)+2\pi\sum_{i=1,i\neq k}^{n}\int_0^{a_i}T_{ik}(r,\rho_0)\,\mathcal{L}\left(\frac{l_{ik,1}^2(r)}{r\rho_0}\right)q_i(\rho_0,\phi)\,\rho_0\,d\rho_0$$

$$=\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_0^r\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)v_k(\rho,\phi). \tag{3.1.26}$$

The next operator to apply is

$$\frac{\mathcal{L}(t)}{t}\frac{d}{dt}\int_t^{a_k}\frac{r dr}{(r^2-t^2)^{1/2}}\mathcal{L}(1/r)$$

which yields

$$-\pi^2 q_k(t,\phi)+2\pi\sum_{i=1,i\neq k}^{n}\int_0^{a_i}\rho_0\,d\rho_0$$

$$\times\left[\frac{\mathcal{L}(t)}{t}\frac{d}{dt}\int_t^{a_k}\frac{r dr}{(r^2-t^2)^{1/2}}T_{ik}(r,\rho_0)\,\mathcal{L}\left(\frac{\rho_0}{l_{ik,2}^2(r)}\right)\right]q_i(\rho_0,\phi)=M_k v_k(t,\phi), \tag{3.1.27}$$

where

$$l_{ik,2}(r)=l_2(r,\rho_0,h_{ik}), \tag{3.1.28}$$

and the M-operator is understood as

$$M_k v_k(t,\phi)=\frac{\mathcal{L}(t)}{t}\frac{d}{dt}\int_r^{a_k}\frac{r dr}{(r^2-t^2)^{1/2}}\mathcal{L}\left(\frac{1}{r^2}\right)\frac{d}{dt}\int_0^r\frac{\rho}{(r^2-\rho^2)^{1/2}}\mathcal{L}(\rho)v_k(\rho,\phi). \tag{3.1.29}$$

Changing the order of integration in (27) and integrating with respect to r, one gets the following system of Fredholm integral equations

$$q_k(\rho,\phi)=-\frac{1}{\pi^2}\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i}K_{ik}(\rho,\rho_0,\phi,\phi_0)q_i(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 -$$

$$-\frac{1}{\pi^2}(M_k v_k)(\rho,\phi), \quad \text{for} \quad k = 1,2,...,n. \tag{3.1.30}$$

Here, the kernel K_{ik} can be expressed in terms of elementary functions:

$$K_{ik}(\rho,\rho_0,\phi,\phi_0) = \frac{h_{ik}}{R_{ik}^3}\left[\frac{R_{ik}}{\xi_{ik}(a_k,\rho)} + \tan^{-1}\frac{\xi_{ik}(a_k,\rho)}{R_{ik}}\right], \tag{3.1.31}$$

$$\xi_{ik}(a_k,\rho) = \frac{(a_k^2 - \rho^2)^{1/2}[a_k^2 - l_{ik,1}^2(a_k)]^{1/2}}{a_k}. \tag{3.1.32}$$

R_{ik} is defined by (23), and h_{ik} by (19). Notice that in the case v_k = const, $M_k v_k(\rho) = v_k(a_k^2 - \rho^2)^{-1/2}$. It is advisable to multiply both sides of (30) by $(a_k^2 - \rho^2)^{1/2}$ in order to eliminate a weak singularity at $\rho \to a_k$.

The system (30) has a unique solution which can be obtained by successive approximations. To prove this, introduce a new function $\sigma_k = (a_k^2 - \rho^2)^{1/2} q_k$. According to the Banach's theorem, it is sufficient to prove that the integral operator

$$W(\sigma_i) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_i}(a_k^2 - \rho^2)^{1/2} K_{ik}(\rho,\rho_0,\phi,\phi_0)\frac{\sigma_i(\rho_0,\phi_0)}{(a_i^2 - \rho_0^2)^{1/2}}\,\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0$$

is a contraction operator. Determine the distance in the class of continuous functions by

$$\delta(\sigma_i,\tilde{\sigma}_i) = \max|\sigma_i(\rho_0,\phi_0) - \tilde{\sigma}_i(\rho_0,\phi_0)|.$$

Assess the value of

$$|W(\sigma_i) - W(\tilde{\sigma}_i)|$$

$$\le \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_i}(a_k^2 - \rho^2)^{1/2} K_{ik}(\rho,\rho_0,\phi,\phi_0)\frac{|\sigma_i(\rho_0,\phi_0) - \tilde{\sigma}_i(\rho_0,\phi_0)|}{(a_i^2 - \rho_0^2)^{1/2}}\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0$$

$$\le \frac{\delta(\sigma_i,\tilde{\sigma}_i)}{\pi^2}\int_0^{2\pi}\int_0^{a_i}(a_k^2 - \rho^2)^{1/2} K_{ik}(\rho,\rho_0,\phi,\phi_0)\frac{\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{(a_i^2 - \rho_0^2)^{1/2}} \le \varepsilon\delta(\sigma_i,\tilde{\sigma}_i), \tag{3.1.33}$$

where

$$\varepsilon = \frac{1}{\pi}\left[\tan^{-1}\frac{a_i + a_k}{h_{ik}} + \tan^{-1}\frac{a_i}{h_{ik}}\right] < 1.$$

The last expression proves that W is a contraction operator thus proving the theorem. The following integral representation was used for the evaluation of the integral in (33)

$$T_{ik}(r,\rho_0) = \frac{h_{ik}}{\pi}\int_0^{\rho_0} \frac{d\tau}{(\rho_0^2 - \tau^2)^{1/2}}\left[\frac{1}{(\tau + r)^2 + h_{ik}^2} + \frac{1}{(\tau - r)^2 + h_{ik}^2}\right]. \tag{3.1.34}$$

After the system (30) is solved, the electrostatic field potential can be defined by

$$V(\rho,\phi,z) = 4\sum_{k=1}^{n}\int_0^{b_k} \frac{dx}{\sqrt{\rho^2 - x^2}}\int_{g_k(x)}^{a_k} \frac{\rho_0 d\rho_0}{(\rho_0^2 - g_k^2(x))^{1/2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) q_k(\rho_0,\phi), \tag{3.1.35}$$

where

$$b_k = l_1(a_k, \rho, z - z_k), \tag{3.1.36}$$

$$g_k(x) = x\sqrt{1 + (z - z_k)^2/(\rho^2 - x^2)}. \tag{3.1.37}$$

The main advantage of (35) over the equivalent expression (17) is its convenience for various mathematical transformations, including the exact evaluation of the integrals as it will be shown further.

The number of unknown densities in (35) can be reduced by substitution of q_k defined by (27) in (35). Using (3), one gets, after simplification

$$V(\rho,\phi,z) = \sum_{i=1, i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i} G_{ik}(\rho,z,\rho_0,\phi - \phi_0)q_i(\rho_0, \phi_0)\,\rho_0 d\rho_0 d\phi_0 + P_k v_k(\rho,\phi), \tag{3.1.38}$$

where G_{ik} can be expressed in elementary functions as follows,

$$G_{ik}(\rho,z,\rho_0,\phi-\phi_0)=\frac{1}{2R_{i,1}}\left[1+\frac{2}{\pi}\tan^{-1}\frac{\eta_{ik,1}}{R_{i,1}}\right]-\frac{1}{2R_{i,2}}\left[1-\frac{2}{\pi}\tan^{-1}\frac{\eta_{ik,2}}{R_{i,2}}\right], \tag{3.1.39}$$

$$R_{i,1}^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z-z_i)^2, \tag{3.1.40}$$

$$R_{i,2}^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z-z_k+z_i-z_k)^2, \tag{3.1.41}$$

$$\eta_{ik,1}=S_{ik}(a_k)+(z-z_k)(z_i-z_k)/S_{ik}(a_k), \tag{3.1.42}$$

$$\eta_{ik,2}=S_{ik}(a_k)-(z-z_k)(z_i-z_k)/S_{ik}(a_k), \tag{3.1.43}$$

$$S_{ik}(a_k)=\frac{\sqrt{l_2^2(a_k,\rho,z-z_k)-a_k^2}\sqrt{l_2^2(a_k,\rho_0,z_i-z_k)-a_k^2}}{a_k}, \tag{3.1.44}$$

and the operator P_k reads

$$P_k v_k(\rho,\phi)=\frac{2}{\pi}\int_0^{a_k} T_k(x,\rho)\,\mathcal{L}\!\left(\frac{\rho}{l_{k,2}^2(x)}\right)\frac{d}{dx}\int_0^x \frac{\rho_0 d\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}(\rho_0)\,v_k(\rho_0,\phi). \tag{3.1.45}$$

Here,

$$T_k(x,\rho)=\frac{1}{(\rho^2-l_1^2(x,\rho,z-z_k))^{1/2}}\frac{\partial}{\partial x}l_1(x,\rho,z-z_k), \tag{3.1.46}$$

$$l_{k,2}(x)=l_2(x,\rho,z-z_k). \tag{3.1.47}$$

In the case $v_k=\text{constant}$

$$P_k v_k=\frac{2}{\pi}v_k\sin^{-1}\frac{a_k}{l_{k,2}(a_k)}. \tag{3.1.48}$$

The solution of some practical problems very often do not require the knowledge of the charge distributions q_k, and only the integral characteristics are of interest, like the total charge, the moments, etc. An assessment of these values can be made without solving the set of equations (30). Indeed, multiplying both sides of (26) by r^m and integrating over the surface of the k-th disk, yields

$$\frac{1}{2}(\pi)^{1/2}\frac{\Gamma[\frac{1}{2}(m+1)]}{\Gamma(1+\frac{1}{2}m)}\int_0^{2\pi}\int_0^{a_k} q_k(\rho_0,\phi_0)\,\rho_0^{m+1}\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0$$

$$+\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i}\left\{\int_0^{a_k} T_{ik}(r,\rho_0)\,r^m\mathrm{d}r\right\}q_i(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{a_k} r^m\,\mathrm{d}r\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{r}\frac{v_k(\rho,\phi)\rho\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}. \tag{3.1.49}$$

It is easy to show that all the intermediary integrals with respect to r in (49) can be evaluated in terms of elementary functions. Expression (49) simplifies for $m=0$ as follows:

$$Q_k+\frac{2}{\pi}\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i} q_i(\rho,\phi)\sin^{-1}\frac{a_k}{l_2(a_k,\rho,h_{ik})}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi=B_k, \tag{3.1.50}$$

where Q_k is the total charge at the k-th disk, and

$$B_k=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_k}\frac{v_k(\rho,\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_k^2-\rho^2}}. \tag{3.1.51}$$

For the case $m=1$, expression (49) reduces to

$$\int_0^{2\pi}\int_0^{a_k} q_k(\rho,\phi)\rho^2\mathrm{d}\rho\mathrm{d}\phi+\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i}\Bigg[\sqrt{\rho^2+h_{ik}^2}-\sqrt{c_{ik}^2-a_k^2}$$

$$+h_{ik}\ln\frac{c_{ik}(\sqrt{c_i^2-a_k^2}+h_{ik}}{\sqrt{c_{ik}^2-a_k^2}(\sqrt{\rho^2+h_{ik}^2}+h_{ik})}\Bigg]q_i(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a_k}\left[\frac{a_k}{\sqrt{a_k^2-\rho^2}}-\cosh^{-1}\frac{a_k}{\rho}\right]v_k(\rho,\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi, \tag{3.1.52}$$

where

$$c_{ik} = l_2(a_k,\rho,h_{ik}).$$

Evoking the mean value theorem which is valid when q_i does not change sign, (50) can be rewritten as:

$$Q_k + \frac{2}{\pi}\sum_{i=1,i\neq k}^{n} Q_i \sin^{-1}\frac{a_k}{l_2(a_k,\rho_{ik},h_{ik})} = B_k \qquad \text{for } k = 1, 2,...,n. \tag{3.1.53}$$

where, according to theorem, $0 \le \rho_{ik} \le a_i$. Despite the fact that the exact value of ρ_{ik} is unknown, the set of linear algebraic equations (53) is very useful for various purposes. For example, one can assess the values of Q_i in the form $Q_{i,\min} \le Q_i \le Q_{i,\max}$ by variation of all the admissible values of ρ_{ik}. In the case of constant potentials v_k, $k = 1,2,...,n$, the set of equations (53) takes the form

$$\frac{\pi}{2a_k}Q_k + \frac{1}{a_k}\sum_{i=1,i\neq k}^{n} Q_i \sin^{-1}\frac{a_k}{l_2(a_k,\rho_{ik},h_{ik})} = v_k \qquad \text{for } k = 1, 2,..., n. \tag{3.1.54}$$

Remembering that the matrix of the capacitances c_{ij} is symmetrical, one can introduce several ways to make the estimation of the values of interest more sharp. For example, let Q_i' be the set of solutions of (54) for $v_{\mathrm{j}} = 1$, and $v_k = 0, k = 1,....,n\ (k \neq j)$. Let $Q_i''\ (i = 1,...,n)$ be the set of solutions of (54) for $v_l = 1$ and all the other $v_k = 0$. Now it is easy to deduce that the following equality should hold

$$c_{lj} = Q_l' = Q_{\mathrm{j}}'' = c_{jl}. \tag{3.1.55}$$

As both quantities are assessed in the form

$$Q'_{l,\max} > Q_l' > Q'_{l,\min}, \qquad Q''_{j,\max} > Q''_j > Q''_{j,\min}, \tag{3.1.56}$$

then the final assessment of c_{lj} will be obtained as an intersection of the intervals (56), which might give a sharper estimation. Here is an illustrative example. Consider a set of two disks with $a_1 = 2$, $a_2 = 1$, $h = 0.6$, $v_1 = v_2 = 1$. The direct estimation of the total charges gives $1.60828 < Q_1 < 2.54544$, $-0.66970 < Q_2 < 0.49519$ while the usage of the described technique allows a much sharper estimation, namely $1.67554 < Q_1 < 1.7801$, $0.27799 < Q_2 < 0.39838$. It is obvious that in the case of equal disks, both intervals in (56) are the same, and their intersection will not improve the estimation. One can expect that in some cases the sharpness of the estimation of certain values for equal disks will be inferior to those of unequal ones.

The set of equations (50) can also be used to verify the accuracy of an approximate solution or a numerical procedure. For example, numerical results for the case of two equal coaxial disks are presented in (Cooke, 1958). Let us check just one point, corresponding to the value $h_{ik}/a_k = 20$. The values, proportional to the total charge Q, given in (Cooke, 1958) for the case of equal potentials and for opposite potentials are $A_0^{(1)} = \frac{1}{2}\pi Q = 0.9683$ and $A_0^{(2)} = 1.0319$ respectively. The estimation of these values by solving (54) is $0.969213 > A_0^{(1)} > 0.969176$, $1.032849 > A_0^{(2)} > 1.032807$ which means that both values given in (Cooke, 1958) are outside the admissible range, and therefore are inaccurate. The correct values are $A_0^{(1)} = 0.969201$ and $A_0^{(2)} = 1.032821$. The numerical procedure to obtain these and other results is discussed further. One can notice also that since $h_{kk} = 0$ and $l_2(a_k,\rho_{kk},0) = a_k$, then the set (53) can be rewritten as:

$$\frac{2}{\pi}\sum_{i=1}^{n} Q_i \sin^{-1}\frac{a_k}{l_2(a_k,\rho_{ik},h_{ik})} = B_k, \qquad \text{for } k = 1, 2, ..., n. \tag{3.1.57}$$

Numerical solution. One needs accurate numerical results in order to estimate the error of various approximate formulae. The collocation method, used here, proved to give an adequate accuracy in a wide range of distances between disks. A case of coaxial disks maintained at constant potential was considered. The set of equations (26) for $v_k = \text{const.}$ takes the form:

$$2\pi\int_r^{a_k}\frac{q_k(\rho_0)\rho_0 d\rho_0}{\sqrt{\rho_0^2 - r^2}} + 2\pi\sum_{i=1, i\neq k}^{n}\int_0^{a_i} T_{ik}(r,\rho_0)q_i(\rho_0)\rho_0 d\rho_0 = v_k. \tag{3.1.58}$$

Let the solution of (58) be presented in the form:

$$q_k(\rho) = (a_k^2 - \rho^2)^{-1/2}\sum_{m=0}^{N} C_{km}\rho^{2m}, \tag{3.1.59}$$

where C_{km} are yet unknown constants. Substituting (59) in (58) and using (34), one gets:

$$\sum_{m=0}^{N} C_{km} f_{km}(r) + \sum_{i=1, i\neq k}^{n} \frac{h_{ik}}{\pi} \int_0^{a_i} \left[\frac{1}{(t+r)^2 + h_{ik}^2} + \frac{1}{(t-r)^2 + h_{ik}^2} \right] \sum_{m=0}^{N} C_{im} f_{im}(t)\, \mathrm{d}t = v_k, \tag{3.1.60}$$

where

$$f_{km}(r) = \int_r^{a_k} \frac{\rho_0^{2m+1}\, \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - r^2}\sqrt{a_k^2 - \rho_0^2}} = \tfrac{1}{2}\pi a_k^{2m} F(-m, \tfrac{1}{2}; 1; 1 - r^2/a_k^2). \tag{3.1.61}$$

is a polynomial in r^2 and F is the Gauss hypergeometric function. All the integrals in (60) can be evaluated exactly and expressed in elementary functions, using the following formulae:

$$\int_0^a \left[\frac{1}{(t+r)^2 + h^2} + \frac{1}{(t-r)^2 + h^2} \right] t^{2n}\, \mathrm{d}t$$

$$= -n r^{2n-1} F\left(1-n, \tfrac{1}{2} - n; \tfrac{2}{3}; -\frac{h^2}{r^2} \right) \ln \frac{(a+r)^2 + h^2}{(a-r)^2 + h^2}$$

$$+ \frac{r^{2n}}{h} F\left(-n, \tfrac{1}{2} - n; \tfrac{1}{2}; -\frac{h^2}{r^2} \right) \left(\tan^{-1} \frac{a+r}{h} + \tan^{-1} \frac{a-r}{h} \right)$$

$$+ \sum_{m=0}^{n-1} \frac{2a^{2n-2m-1} r^{2m}}{2n-2m-1} (2m+1) F\left(-m, \tfrac{1}{2} - m; \tfrac{3}{2}; -\frac{h^2}{r^2} \right), \tag{3.1.62}$$

$$\int_0^a \left[\frac{1}{(t+r)^2 + h^2} + \frac{1}{(t-r)^2 + h^2} \right] t^{2n+1}\, \mathrm{d}t$$

$$= r^{2n} \Bigg\{ (2n+1) F\left(-n, \tfrac{1}{2} - n; \tfrac{3}{2}; -\frac{h^2}{r^2} \right) \ln \frac{l_2^2 - l_1^2}{r^2 + h^2}$$

$$+ \frac{r^{2n+1}}{h} F\left(-n, \tfrac{1}{2} - n; \tfrac{1}{2}; -\frac{h^2}{r^2} \right) \left(2\tan^{-1} \frac{r}{h} + \tan^{-1} \frac{a-r}{h} - \tan^{-1} \frac{a+r}{h} \right) \Bigg\}$$

$$+\sum_{m=0}^{n-1} \frac{2m+1}{n-m} a^{2n-2m} r^{2m} F\left(-m, \tfrac{1}{2}-m; \tfrac{3}{2}; -\frac{h^2}{r^2}\right), \qquad (3.1.63)$$

where l_1 and l_2 are understood as $l_{1,2}(a,r,h)$ defined by (1).

Now one has to specify $N+1$ points of collocation at each of the intervals $0 \le r \le a_k$, $k = 1,2,...,n$, and request that the set of equations (60) be satisfied at these points. This leads to the set of $n(N+1)$ linear algebraic equations:

$$\sum_{m=0}^{N} C_{km} f_{km}(r_j) + \sum_{i=1, i\neq k}^{n} \frac{h_{ik}}{\pi} \sum_{m=0}^{N} C_{im} \int_0^{a_i} \left[\frac{1}{(t+r_j)^2 + h_{ik}^2} + \frac{1}{(t-r_j)^2 + h_{ik}^2} \right] f_{im}(t)\, dt = v_k,$$

$$\text{for } j = 0,1,...,N \quad \text{and} \quad k = 1,2,...,n. \qquad (3.1.64)$$

from which the constants C_{km} are to be defined. After the system (64) is solved, all the other parameters of interest can be obtained rather easily, for example, the total charge at each disk can be defined by

$$Q_k = \sum_{m=0}^{N} \tfrac{1}{2}(\pi)^{1/2} \frac{\Gamma(m+1)}{\Gamma(m+\tfrac{3}{2})} a_k^{2m+1} C_{km}. \qquad (3.1.65)$$

The potential value at an arbitrary point in space can also be expressed in terms of elementary functions. Indeed, substitution of (59) in (35) gives, after the first integration,

$$V(\rho,\phi,z) = 4 \sum_{k=1}^{n} \int_0^{b_k} \frac{dx}{\sqrt{\rho^2 - x^2}} \sum_{m=0}^{N} C_{km} f_{km}(g_k(x)). \qquad (3.1.66)$$

Since f_{km} here is a polynomial, according to (61), then the remaining integrals can always be evaluated exactly in terms of elementary functions. The accuracy of the solution was assessed by the error function $E_k(r)$ defined by:

$$E_k(r) = \sum_{m=0}^{N} C_{km} f_{km}(r)$$

$$+\sum_{i=1,i\neq k}^{n}\frac{h_{ik}}{\pi}\sum_{m=0}^{N}C_{im}\int_{0}^{a_i}\left[\frac{1}{(t+r)^2+h_{ik}^2}+\frac{1}{(t-r)^2+h_{ik}^2}\right]f_{im}(t)\,\mathrm{d}t-v_k$$

for $0\le r\le a_k,\quad k=1,2,...,n.$ (3.1.67)

It is obvious that $E_k=0$ indicates that the solution is exact. The value of

$$\Delta=\max|E_k(r)| \qquad (3.1.68)$$

was used as a measure of accuracy of the solution, which means that, out of two solutions, the one with the smallest Δ was considered more accurate. The typical behavior of the error function E_k is presented in Fig. 3.1.1 and Fig. 3.1.2 for the case of two equal coaxial disks of radius 1 held at unit opposite potentials and a gap between them $h=0.1$. The solid line in both

Fig. 3.1.1. Error function for equidistant points of collocations.

figures plots the error function for the case of three points of collocations, the dashed line gives the same plot for five points of collocations, and the error function for 11 points of collocations is given by circles. Fig. 3.1.1 corresponds to the case of equidistant points while in Fig. 3.1.2 the points of collocations were taken at $r_j=a_k\sin(\pi j/2N)$, $j=0,1,...,N$. Comparison of figures leads to a conclusion that the accuracy for the case of equidistant points of collocation is

Fig. 3.1.2. Error function for non-equidistant points of collocations.

inferior to the second choice. Our investigation also showed that further increase in the number of points of collocation generally does not improve the accuracy of the solution, and in many cases the accuracy deteriorates. The error of evaluation of the total charge Q_k was taken as a product $Q_k\Delta$. The real error is unknown but it will definitely be less than $Q_k\Delta$ due to the fluctuation of the error function with the area under the positive half-wave being almost equal to the negative area.

The results of the numerical procedure with 11 points of collocation $r_j = \sin(\pi j/20)$, $j = 0, 1, ..., 10$, for the case of two equal coaxial disks of radius 1, held at equal and at opposite potentials with a variable gap h between them, are given in Table 3.1.1. The value of $Q^* = Q_k\pi/2v_k a_k$ along with the absolute error assessment is given in Table 3.1.1. Table 3.1.1 corrects some inaccuracies in similar results published in (Cooke, 1958).

Figure 3.1.3 plots the charge density distribution for different values of the gap h. Several examples are considered below.

Two equal coaxial disks. Denote $a_1 = a_2 = a$; $h_{12} = h_{21} = h$. Two fundamental cases are considered: $v_1 = v_2 = v$ and $v_1 = -v_2 = v$. The solution of the set of equations (54) has the form

Table 3.1.1. Values of the total charge.

h/a	Equal potentials		Opposite potentials	
	Q^*	Error	Q^*	Error
0	0.5	0	∞	0
0.001	0.500877	0.002	789.29	2
0.01	0.505320	0.0015	80.457	0.3
0.05	0.520553	0.0010	17.22936	0.03
0.1	0.535883	0.0002	9.233071	0.001
0.2	0.561362	0.00005	5.175753	0.0001
0.4	0.602499	10^{-7}	3.102305	10^{-5}
0.6	0.636407	10^{-8}	2.395441	10^{-6}
0.8	0.665610	10^{-9}	2.037267	10^{-7}
1.0	0.691207	10^{-10}	1.820785	10^{-8}
1.2	0.713812	10^{-11}	1.676043	10^{-9}
1.5	0.743019	10^{-11}	1.531444	10^{-10}
2.0	0.781752	10^{-11}	1.388027	10^{-11}
2.5	0.811259	10^{-11}	1.303422	10^{-11}
3.0	0.834216	10^{-12}	1.248107	10^{-11}
5.0	0.889579	10^{-13}	1.141723	10^{-12}
10.0	0.940518	10^{-15}	1.067514	10^{-13}
20.0	0.969201	10^{-15}	1.032821	10^{-15}
100.0	0.993674	10^{-16}	1.006406	10^{-15}
∞	1.	0	1.	0

Fig. 3.1.3. Charge density distribution (two disks at unit opposite potentials).

$$Q_1^{\pm} = \frac{2}{\pi} \frac{av}{1 \pm \frac{2}{\pi} \sin^{-1} [a/l_2(a,\rho^{\pm},h)]}. \tag{3.1.69}$$

The plus sign in (69) corresponds to the case of equal potentials, and the minus to opposite ones. Notice that formula (69) gives exact results for $h \to 0$ and $h \to \infty$. Considering Q_1^+ and Q_1^- as functions of ρ^+ and ρ^- respectively, one can analyze numerically the overall performance of (69). The simplest approximate formula can be obtained by averaging of the maximum and the minimum admissible values of $Q_1^{\pm}$, namely

$$Q_1^+ = \tfrac{1}{2}[Q_1^+(a) + Q_1^+(0)], \qquad Q_1^- = \tfrac{1}{2}[Q_1^-(a) + Q_1^-(0)]. \tag{3.1.70}$$

Comparison with results of Table 1 shows that the maximum error of the first formula (70) is about 3%, while the second one yields about 12%. Better accuracy can be obtained by assuming

$$Q_1^+ = \tfrac{1}{2}[Q_1^+(0.67a) + Q_1^+(0.98a)], \qquad Q_1^- = \tfrac{1}{2}[Q_1^-(0.562a) + Q_1^-(0.983a)]. \tag{3.1.71}$$

The maximum error is less than 0.1% for the first formula of (71), and is about 0.62% for the second one.

If the accuracy achieved is still not satisfactory, one has to analyze ρ^+ and ρ^- as functions of h. The physical meaning of $\rho^{\pm}$ can be explained as a substitution of a disk by an infinitely thin annulus of radius $\rho^{\pm}$, having the same total charge and an equivalent influence on the total charge of the second disk. Plotting of both curves $\rho^+ = \rho^+(h)$ and $\rho^- = \rho^-(h)$ can help also to verify the accuracy of a numerical procedure. Elementary logic suggests that both should be smooth curves merging as $h \to \infty$.

Using (69) and Table 1, the curves $\rho^+(h)$ and $\rho^-(h)$ were plotted in Fig. 3.1.4 by the dashed line and the solid line respectively. The limiting values were established as follows:

$$\rho^+(0) = a, \quad \rho^-(0) = \sqrt{3/4}\,a, \quad \rho^+(\infty) = \rho^-(\infty) = \sqrt{2/3}\,a. \tag{3.1.72}$$

Similar computations were made by using the data from (Cooke, 1958). The results for ρ^+ and ρ^- are presented in Fig. 3.1.4 by non-solid circles and solid circles respectively. Looking at Fig. 3.1.4, we note immediately that there are some troubles with the accuracy of the data given by Cooke. We note also that the results of Cooke for $h=5$ and $h=10$ are not incorrect, they just do not

Fig. 3.1.4. Test of the accuracy of numerical results

have sufficient number of decimal places in the data, and this indicates how sensitive the parameter $\rho^{\pm}$ is for large h. No reasonable values for $\rho^{\pm}$ can be obtained from Cooke's data for $h=20$. As was shown earlier, these data are beyond the admissible interval.

One can approximate the function $\rho^{-}(h)$ as

$$\rho^{-} = (-0.02347\,e^{-0.036h} + 0.073\,e^{-8.7h} + \sqrt{2/3})a. \tag{3.1.73}$$

Substitution of (73) in (69) makes it highly accurate in the whole range $0<h<\infty$, with the maximum error not exceeding 0.3%.

Three equal coaxial disks at constant potentials. We consider the case of equal disks because it is the least favorable case in terms of accuracy, as it was stated in previous section. Put $a_1=a_2=a_3=a$, $h_{12}=h_{23}=h$, $h_{13}=2h$. The set of equations to be solved is

$$\frac{\pi}{2a}Q_1 + Q_2\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{21},h)} + Q_3\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{31},2h)} = v_1,$$

$$Q_1\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{21},h)} + \frac{\pi}{2a}Q_2 + Q_3\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{23},h)} = v_2,$$

$$Q_1 \frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{31},2h)} + Q_2 \frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{23},h)} + \frac{\pi}{2a}Q_3 = v_3. \tag{3.1.74}$$

Solution of (74) for all the combinations of ρ_{21}, ρ_{31} and ρ_{23} equal 0 or a gives the upper and the lower bound for the values of total charge Q_1, Q_2 and Q_3. The problem of three equal, equally spaced disks was considered in (Kuz'min,1971). The following approximate solution was given there in the assumption that $h > a$.

$$\begin{aligned} Q_1 &= v_1(c_{11} + c_{12} + c_{13}) - v_2 c_{12} - v_3 c_{13},\\ Q_2 &= -v_1 c_{12} + v_2(c_{22} + c_{21} + c_{23}) - v_3 c_{23},\\ Q_3 &= -v_1 c_{31} - v_2 c_{32} + v_3(c_{33} + c_{31} + c_{32}),\end{aligned} \tag{3.1.75}$$

where

$$\begin{aligned} c_{11} &= c_{33} = 0.6366a(1 - 0.9549\varepsilon + 1.1145\varepsilon^2 - 0.6514\varepsilon^3 - 0.02069\varepsilon^4),\\ c_{22} &= 0.6366a(1 - 1.2732\varepsilon + 1.2159\varepsilon^2 - 0.5702\varepsilon^3 + 0.01874\varepsilon^4),\\ c_{12} &= c_{21} = c_{32} = c_{23} = 0.4053a\varepsilon(1 - 0.3183\varepsilon + 0.2452\varepsilon^2 - 0.2830\varepsilon^3),\\ c_{31} &= c_{13} = 0.2026a\varepsilon(1 - 1.2732\varepsilon + 0.7452\varepsilon^2 + 0.2786\varepsilon^3), \quad \varepsilon = a/h.\end{aligned} \tag{3.1.76}$$

It is of interest to compare the results of solution of (74) with those of (75) and (76). Necessary calculations were performed for two particular cases: $v_1 = v_2 = v_3 = v$ and $v_1 = -v_2 = v$, $v_3 = 0$. The results of evaluation of the dimensionless $Q_1^* = \pi Q_1/(2va)$ are presented in Fig. 3.1.5 and Fig. 3.1.6 respectively. The solid line in both figures gives the upper bound, the dashed line gives the lower bound, (computed from (74)) and the results of (75) are plotted by circles. We can see that formulae (75) and (76) give good accuracy for $h/a > 2$; the results sharply deviate from the admissible region for $h/a < 1.5$.

Discussion. It is of interest to establish a relationship between some of the results of this section and those previously reported in literature. Expression (39) corresponds to the Green's function for a conducting disk under the influence of a charged point, found by Hobson (1900). His expression in our notation takes the form

$$G_{ik} = \frac{1}{2R_{i,1}}\left[1 \pm \frac{2}{\pi}\tan^{-1}\frac{\sqrt{\eta_{ik,1}^2}}{R_{i,1}}\right] - \frac{1}{2R_{i,2}}\left[1 \mp \frac{2}{\pi}\tan^{-1}\frac{\sqrt{\eta_{ik,2}^2}}{R_{i,2}}\right],$$

where the ambiguous signs are assigned according to pretty complicated rules

Fig. 3.1.5. Total charge at the first disk (three-disk system $v_1 = v_2 = v_3 = v$)

Fig. 3.1.6. Total charge at the first disk (three-disk system $v_1 = -v_2 = v$, $v_3 = 0$)

depending on the position of the points. The geometrical form, in which Hobson presented his result, did not let him to notice the square root of a complete square which led to the ambiguous signs. Our expressions (39) is simpler and free of this mishap. Expression (31) corresponds to another source function, also found by Hobson in geometrical form.

Certain relationship can be established between the set of equations (26) and the Love type integral equations derived in (Kuz'min, 1971). Introduce a new unknown function

$$\chi_k(r,\phi) = 2\pi \int_r^{a_k} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - r^2}} \mathcal{L}\left(\frac{r}{\rho_0}\right) q_k(\rho_0,\phi), \quad k=1, 2,..., n. \tag{3.1.77}$$

Inversion of (77) gives

$$q_k(\rho_0,\phi) = -\frac{\mathcal{L}(\rho_0)}{\pi^2 \rho_0} \frac{\mathrm{d}}{\mathrm{d}\rho_0} \int_{\rho_0}^{a_k} \frac{x \mathrm{d}x}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{1}{x}\right) \chi_k(x,\phi). \tag{3.1.78}$$

Substitution of (77) and (78) in (26) yields, after the change of the order of integration

$$\chi_k(r,\phi) + \frac{2}{\pi} \sum_{i=1, i\neq k}^{n} \int_0^{a_i} \left\{ \mathcal{L}\left(\frac{1}{x}\right) \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x T_{ik}(r,\rho_0) \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{l_{ik,1}^2(r)}{r}\right) \right\} \chi_i(x,\phi)\, \mathrm{d}x$$

$$= \mathcal{L}\left(\frac{1}{r}\right) \frac{\mathrm{d}}{\mathrm{d}r} \int_0^r \frac{\rho \mathrm{d}\rho}{(r^2 - \rho^2)^{1/2}} \mathcal{L}(\rho) v_k(\rho,\phi). \tag{3.1.79}$$

The internal integral with respect to ρ_0 in (79) can be evaluated and expressed in terms of elementary function for each particular harmonic. For example, in the case of axial symmetry one gets from (79)

$$\chi_{k0}(r) + \frac{1}{\pi} \sum_{i=1, i\neq k}^{n} h_{ik} \int_0^{a_i} \left[\frac{1}{(x+r)^2 + h_{ik}^2} + \frac{1}{(x-r)^2 + h_{ik}^2} \right] \chi_{i0}(x)\, \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}r} \int_0^r \frac{v_{k0}(\rho) \rho \mathrm{d}\rho}{(r^2 - \rho^2)^{1/2}}. \tag{3.1.80}$$

The set of equations (80) corresponds to the one derived in (Kuz'min, 1971). Here, χ_{k0} and v_{k0} are the zero harmonics of the functions χ_k and v_k respectively. In general, for the m-th harmonic the following integral equation can be obtained from (79)

$$\chi_{km}(r)+\frac{2}{\pi}\sum_{i=1,i\neq k}^{n}h_{ik}\int_0^{a_i}H_{ikm}(r,x)\chi_{im}(x)\,\mathrm{d}x=\frac{1}{r^m}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{r}\frac{v_{km}(\rho)\rho^{m+1}\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}},$$

for $k=0,1,...,n$. (3.1.81)

The kernel H_{ikm} can be expressed in elementary functions:

$$H_{ikm}(r,x)=\left(\frac{r}{x}\right)^m\left\{\frac{2x^2}{(l_2^2-l_1^2)^2}-\sum_{k=1}^{m-1}\frac{(-1)^k}{\Gamma(k)}\left(\frac{l_1}{r}\right)^{2k}\left[\frac{x^2-l_2^2}{l_2^2}\frac{\partial^{k-1}}{\partial\tau^{k-1}}((1-\tau)^{k-1}\mu(\tau))\right.\right.$$

$$\left.\left.+\frac{x^2(l_1^2+l_2^2-2r^2)}{l_2^2(l_2^2-l_1^2)}\frac{\partial^{k-1}}{\partial\tau^{k-1}}\left((1-\tau)^k\frac{\partial\mu(\tau)}{\partial\tau}\right)\right]\frac{1}{l_2^2-l_1^2}\right\}$$

for $m\geq 1$. (3.1.82)

where l_1 and l_2 are understood as $l_{1,2}(r,x,h_{ik})$, and

$$\mu(\tau)=\frac{1}{\sqrt{\tau}}\ln\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}},\qquad \tau=\frac{l_1^2(r,x,h_{ik})}{l_2^2(r,x,h_{ik})}. \tag{3.1.83}$$

Here are explicit expressions of the kernel for the first six harmonics

$$H_{ik,1}(r,x)=\frac{1}{2}\left[\frac{1}{(r-x)^2+h_{ik}^2}-\frac{1}{(r+x)^2+h_{ik}^2}\right]=\frac{2l_1l_2}{(l_2^2-l_1^2)^2},$$

$$H_{ik,2}(r,x)=\frac{1}{2}\left[\frac{1}{(r-x)^2+h_{ik}^2}+\frac{1}{(r+x)^2+h_{ik}^2}-\frac{1}{2rx}\ln\frac{(r+x)^2+h_{ik}^2}{(r-x)^2+h_{ik}^2}\right]$$

$$=\frac{l_2^2+l_1^2}{(l_2^2-l_1^2)^2}-\frac{1}{2l_1l_2}\ln\frac{l_2+l_1}{l_2-l_1},$$

$$H_{ik,3}(r,x)=\frac{1}{2}\left[\frac{1}{(r-x)^2+h_{ik}^2}-\frac{1}{(r+x)^2+h_{ik}^2}+\frac{3}{rx}\right.$$

$$\left.-\frac{3(r^2+x^2+h_{ik}^2)}{4r^2x^2}\ln\frac{(r+x)^2+h_{ik}^2}{(r-x)^2+h_{ik}^2}\right]$$

$$=\frac{2l_1l_2}{(l_2^2-l_1^2)^2}+\frac{3}{l_1l_2}-\frac{3(l_2^2+l_1^2)}{4l_1^2l_2^2}\ln\frac{l_2+l_1}{l_2-l_1},$$

$$H_{ik,4}(r,x)=\frac{l_2^2+l_1^2}{(l_2^2-l_1^2)^2}+\frac{15(l_2^2+l_1^2)}{8l_1^2l_2^2}-\frac{3(5l_2^4+6l_1^2l_2^2+5l_1^4)}{16l_1^3l_2^3}\ln\frac{l_2+l_1}{l_2-l_1},$$

$$H_{ik,5}(r,x)=\frac{105l_2^8-40\,l_1^2l_2^6-34l_1^4l_2^4-40l_1^6l_2^2+105l_1^8}{48l_1^3l_2^3(l_2^2-l_1^2)^2}$$

$$-\frac{5(l_2^2+l_1^2)(7l_2^4+2l_1^2l_2^2+7l_1^4)}{32l_1^4l_2^4}\ln\frac{l_2+l_1}{l_2-l_1},$$

$$H_{ik,6}(r,x)=\frac{(l_2^2+l_1^2)(315l_2^8-420l_1^2l_2^6+338l_1^4l_2^4-420l_1^6l_2^2+315l_1^8)}{128l_1^4l_2^4(l_2^2-l_1^2)^2}$$

$$-\frac{15(21l_2^8+28l_1^2l_2^6+30l_1^4l_2^4+28l_1^6l_2^2+21l_1^8)}{256l_1^5l_2^5}\ln\frac{l_2+l_1}{l_2-l_1}, \qquad (3.1.84)$$

We recall once again that in this section, unlike elsewhere in the book, the abbreviations l_1 and l_2 denote $l_1(r,x,h_{ik})$ and $l_2(r,x,h_{ik})$ respectively. The kernels defined by (82) contain no singularities therefore the regular methods of solution of Fredholm integral equations are applicable here. Formulae (81)-(84) seem not to have been reported in literature before.

3.2. Potential of arbitrarily located disks

The electrostatic field of several non-parallel circular disks is considered. A set of governing integral equations is derived by a new method. It is shown that some integral characteristics can be found without solving the integral equations. The upper and the lower bounds for the total charge are found from a set of linear algebraic equations whose coefficients are defined by simple geometric characteristics of the system. Example considered shows sufficient sharpness of the estimations.

There are just a few papers where the problem of two *coplanar* disks is considered; among them we know of only one (Kobayashi 1939) where some numerical results of sufficient accuracy are given. A solution to the problem of two *non-parallel* disks, whose centers are located in one plane orthogonal to the planes of both disks, can be found in Ufliand (1977). It was solved by the Mehler-Fok transform with consequent use of the small parameter method. To the best of our knowledge there are no publications considering the electrostatic problem of two or more *arbitrarily located* disks, mainly due to the fact that existing methods are not capable of solving these problems.

We consider a system of n charged arbitrarily located circular disks. Let a_i be the radius of the i-th disk, and S_i be its surface. We can single out, without loss of generality, disk number one and place the origin of the set of cylindrical coordinates (ρ,ϕ,z) at its center so that the Oz axis is orthogonal to the disk's plane. Let the position vector $\mathbf{r}_i$ indicate the center of the i-th disk, and the unit vector $\mathbf{n}_i$, orthogonal to the disk's plane, indicate its orientation. The problem is to find the electrostatic potential due to the system of charged disks, i.e. to find a harmonic function $V(\rho,\phi,z)$, satisfying the following boundary conditions:

$$V(\rho,\phi,z) = v_i(\rho,\phi,z) \quad \text{for} \quad (\rho,\phi,z)\subset S_i; \quad i = 1,2,...,n. \tag{3.2.1}$$

The potential can be represented by a simple layer distribution as follows

$$V = \sum_{i=1}^{n} \int\int_{S_i} \frac{q_i}{R_i} \mathrm{d}S_i. \tag{3.2.2}$$

Here q_i are the as yet unknown charge densities, and R_i stands for the distance between a point of integration inside S_i and an arbitrary point in space.

Now make use of the following integral representation for the reciprocal distance (see (1.2.19))

$$\frac{1}{\sqrt{\rho^2+\rho_i^2-2\rho\rho_i\cos(\phi-\phi_i)+z_i^2}} = \frac{2}{\pi}\int_0^{c_i(\rho)} \frac{\lambda\left(\dfrac{x^2}{\rho\rho_i}\,\phi-\phi_i\right)\mathrm{d}x}{\sqrt{\rho_i^2-x^2}\,\sqrt{\rho^2-g_i^2(x)}}, \tag{3.2.3}$$

where

$$g_i^2(x) = x^2\left[1+\frac{z_i^2}{\rho_i^2-x^2}\right], \tag{3.2.4}$$

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}, \tag{3.2.5}$$

and

$$c_i(\rho) = \frac{1}{2}\{[(\rho+\rho_i)^2+z_i^2]^{1/2} - [(\rho-\rho_i)^2+z_i^2]^{1/2}\}. \tag{3.2.6}$$

An obvious simplification of (3) is valid when $z_i = 0$, namely

$$\frac{1}{\sqrt{\rho^2+\rho_i^2-2\rho\rho_i\cos(\phi-\phi_i)}}=\frac{2}{\pi}\int_0^{\min(\rho,\rho_i)}\frac{\lambda\left(\frac{x^2}{\rho\rho_i}\phi-\phi_i\right)\mathrm{d}x}{\sqrt{\rho_i^2-x^2}\sqrt{\rho^2-x^2}}, \tag{3.2.7}$$

Substituting the boundary conditions (1) for the first disk into (2) and using (3) and (7) yields the following integral equation

$$4\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^{a_1}\frac{r\,\mathrm{d}r}{\sqrt{r^2-x^2}}\mathcal{L}\left(\frac{x^2}{\rho r}\right)q_1(r,\phi)$$

$$+\frac{2}{\pi}\sum_{i=2}^{n}\int\int_{S_i}\left\{\int_0^{c_i(\rho)}\frac{\lambda\left(\frac{x^2}{\rho\rho_i}\phi-\phi_i\right)\mathrm{d}x}{\sqrt{\rho_i^2-x^2}\sqrt{\rho^2-g_i^2(x)}}\right\}q_i\,\mathrm{d}S_i=v_1(\rho,\phi). \tag{3.2.8}$$

Let us apply the operator

$$\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\mathcal{L}(\rho)$$

to both sides of (8). The result of application of the operator above is

$$2\pi\int_y^{a_1}\frac{r\,\mathrm{d}r}{\sqrt{r^2-y^2}}\mathcal{L}\left(\frac{y}{r}\right)q_1(r,\phi)+\sum_{i=2}^{n}\int\int_{S_i}\frac{\lambda[c_i^2(y)/y\rho_i,\,\phi-\phi_i]}{\sqrt{\rho_i^2-c_i^2(y)}}\frac{\mathrm{d}c_i(y)}{\mathrm{d}y}q_i\,\mathrm{dS}_i$$

$$=\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\mathcal{L}(\rho)\,v_1(\rho,\phi). \tag{3.2.9}$$

Here the following rule for the interchange of the order of integration was used

$$\int_0^{y}\mathrm{d}\rho\int_0^{c_i(\rho)}\mathrm{d}x=\int_0^{c_i(y)}\mathrm{d}x\int_{g_i(x)}^{y}\mathrm{d}\rho. \tag{3.2.10}$$

One can easily notice that each function $g(x)$ is inverse to the relevant function $c(\rho)$. The next operator to apply is

$$\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\mathcal{L}\left(\frac{1}{y}\right)$$

with the result

$$q_1(t,\phi)-\frac{1}{\pi^2}\sum_{i=2}^{n}\int\!\!\int_{S_i}\left[\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\frac{\lambda[c_i^2(y)/(y^2\rho_i),\phi-\phi_i]}{\sqrt{\rho_i^2-c_i^2(y)}}\frac{\mathrm{d}c_i(y)}{\mathrm{d}y}\right]q_i\,\mathrm{d}S_i$$

$$=-\frac{1}{\pi^2}\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\mathcal{L}\left(\frac{1}{y^2}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\,\mathcal{L}(\rho)v_1(\rho,\phi). \qquad (3.2.11)$$

Integration with respect to y can be performed in (11) (see (1.3.32)) to give

$$q_1(t,\phi)=-\frac{1}{\pi^2}\sum_{i=2}^{n}\int\!\!\int_{S_i}K_{1i}(t,\phi,\rho_i,\phi_i,z_i)\,q_i\,\mathrm{d}S_i+\frac{1}{\pi^2}M_1v_1(t,\phi), \qquad (3.2.12)$$

where the kernel can be expressed in elementary functions

$$K_{1i}(t,\phi,\rho_i,\phi_i,z_i)=\frac{|z_i|}{R_{1i}^3}\left[\frac{R_{1i}}{\xi_{1i}}+\tan^{-1}\frac{\xi_{1i}}{R_{1i}}\right],$$

$$\xi_{1i}=\sqrt{a_1^2-t^2}\;\;\sqrt{a_1^2-c_i^2(a_1)}/a_1,$$

$$R_{1i}=[t^2+\rho_i^2-2\,t\rho_i\cos(\phi-\phi_i)+z_i^2]^{1/2},$$

and

$$M_1(t,\phi)=\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\mathcal{L}\left(\frac{1}{y^2}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\,\mathcal{L}(\rho)v_1(\rho,\phi). \qquad (3.2.13)$$

Similar equations can be derived for the other disks thus forming a set of integral equations to be solved. One has to remember that each such equation is valid in the local set of coordinates related to the particular disk. It is also important to notice that during the derivation we only used the assumption that S_1 was a circular disk, equation (12) would remain unchanged if S_i (i>1) were arbitrary surfaces. It is possible to prove (see section 3.1) that the set of equations (12) can be solved by successive iterations but the most interesting feature of these equations is the ability to obtain the estimation for some integral characteristics without solving the equations. For example, the estimation of the total charge can be made in the following manner. Multiplying both sides of (11) by $t\,\mathrm{d}t\,\mathrm{d}\phi$ and integrating over the surface of the first disk, one gets

$$Q_1 + \frac{2}{\pi}\sum_{i=2}^{n}\int\!\!\int_{S_i} \sin^{-1}\left(\frac{c_i(a_1)}{\rho_i}\right) q_i \,\mathrm{d}S_i = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_1} \frac{v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_1^2-\rho^2}}. \qquad (3.2.14)$$

Introducing a new quantity $b_i(\rho)$ as

$$b_i(\rho) = \frac{1}{2}\{[(\rho+\rho_i)^2 + z_i^2]^{1/2} + [(\rho-\rho_i)^2 + z_i^2]^{1/2}\} \qquad (3.2.15)$$

with an obvious property $c_i(\rho)\,b_i(\rho) = \rho\rho_i$, expression (14) can be rewritten in the form

$$Q_1 + \frac{2}{\pi}\sum_{i=2}^{n}\int\!\!\int_{S_i} \sin^{-1}\left(\frac{a_1}{b_i(a_1)}\right) q_i \,\mathrm{d}S_i = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_1} \frac{v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_1^2-\rho^2}}. \qquad (3.2.16)$$

Evoking the mean value theorem which is valid when q_i does not change sign, expression (14) can be evaluated as follows

$$Q_1 + \frac{2}{\pi}\sum_{i=2}^{n} Q_i \sin^{-1}\left(\frac{a_1}{b_{i1}}\right) = B_1, \qquad (3.2.17)$$

where Q_i stands for the total charge on the i-th disk, and

$$B_1 = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_1} \frac{v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_1^2-\rho^2}}, \qquad (3.2.18)$$

$$b_{i1} = \frac{1}{2}\{[(a_1+\rho_{i1})^2 + z_{i1}^2]^{1/2} + [(a_1-\rho_{i1})^2 + z_{i1}^2]^{1/2}\}. \qquad (3.2.19)$$

The physical meaning of b_{i1} is quite obvious: it represents a half of the sum of distances from a point inside S_i to the closest and the farthermost points of the first disk's edge.

Equation similar to (17) can be derived for the other disks, and the following set of linear algebraic equations with respect to the total charges Q_i can be written

$$Q_k + \frac{2}{\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} Q_i \sin^{-1}(a_k/b_{ik}) = B_k, \quad \text{for} \quad k = 1, 2, 3, ..., n; \tag{3.2.20}$$

where

$$B_k = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_k} \frac{v_k(\rho,\phi)\,\rho\,d\rho\,d\phi}{\sqrt{a_k^2-\rho^2}}, \tag{3.2.21}$$

$$b_{ik} = \frac{1}{2}\{[(a_k+\rho_{ik})^2+z_{ik}^2]^{1/2} + [(a_k-\rho_{ik})^2+z_{ik}^2]^{1/2}\}. \tag{3.2.22}$$

Of course, the exact values of ρ_{ik} and z_{ik} are not known but the fact that $(\rho_{ik}, z_{ik}) \subset S_i$ allows us to obtain the upper and the lower bounds for the total charges by solving the set (20) for the extreme points. It will be shown later that this estimation is sufficiently sharp and can be used for verification of the accuracy of various approximate solutions. Notice also that in the case $v_k(\rho,\phi) = v_k = \text{const.}$,

$$M_k v_k(\rho,\phi) = v_k/\sqrt{a_k^2-\rho^2}, \quad B_k = \frac{2}{\pi} v_k a_k. \tag{3.2.23}$$

Since $b_{kk} = a_k$, the set of equations (20) can be rewritten in a uniform manner

$$\frac{2}{\pi}\sum_{i=1}^{n} Q_i \sin^{-1}(a_k/b_{ik}) = B_k, \quad \text{for} \quad k = 1, 2, 3, ..., n. \tag{3.2.24}$$

The possibility to assess the integral characteristics in such a simple manner is not limited to the quantity of total charge. One can multiply (11) by $t^m dt\,d\phi$ and integrate over the surface S_1. The result can always be expressed in elementary functions. For example, in the case $m=2$, the result of integration is

$$\int_0^{2\pi}\int_0^{a_1} q_1(t,\phi)\, t^2\, dt\, d\phi + \sum_{i=2}^{n}\int\int_{S_i}\Big[\sqrt{\rho_i^2+z_i^2} \;-\sqrt{b_i^2(a_1)-a_1^2}$$

$$+ |z_i| \ln \frac{b_i(a_1)\,[\sqrt{b_i^2(a_1) - a_1^2} + |z_i|]}{\sqrt{b_i^2(a_1) - a_1^2}[|z_i| + \sqrt{\rho_i^2 + z_i^2}]}\Bigg] q_i \, \mathrm{d}S_i$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{a_1} \left[\frac{a_1}{\sqrt{a_1^2 - \rho^2}} - \cosh^{-1}\left(\frac{a_1}{\rho}\right) \right] v_1(\rho, \phi)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi.$$

Here, one can again evoke the mean value theorem and get the upper and the lower bounds for the quantities of interest.

Example 1. The simplest example to consider is the case of two disks of radii R_1 and R_2 lying in two planes intersecting at an angle α and whose centers are lying in one plane orthogonal to the line of intersection at the distances d_1 and d_2 from the line. Let the disks be conductors charged to the potentials V_1 and V_2 respectively. The total charges Q_1 and Q_2 are to be determined. The set of equations to be solved has the form

$$Q_1 + \frac{2}{\pi} Q_2 \sin^{-1}\left(\frac{a_1}{b_{21}}\right) = \frac{2}{\pi} V_1 a_1$$

$$\frac{2}{\pi} Q_1 \sin^{-1}\left(\frac{a_2}{b_{12}}\right) + Q_2 = \frac{2}{\pi} V_2 a_2. \tag{3.2.25}$$

Here

$$b_{12} = \frac{1}{2} \{[(d_2 + x)^2 + (d_1 - R_1)^2 - 2(d_2 + x)(d_1 - R_1)\cos\alpha]^{1/2}$$

$$+ [(d_2 + x)^2 + (d_1 + R_1)^2 - 2(d_2 + x)(d_1 + R_1)\cos\alpha]^{1/2}\},$$

$$b_{21} = \frac{1}{2} \{[(d_1 + y)^2 + (d_2 - R_2)^2 - 2(d_1 + y)(d_2 - R_2)\cos\alpha]^{1/2}$$

$$+ [(d_1 + y)^2 + (d_2 + R_2)^2 - 2(d_1 + y)(d_2 + R_2)\cos\alpha]^{1/2}\}, \tag{3.2.26}$$

where $-R_2 \le x \le R_2$, and $-R_1 \le y \le R_1$. The extreme points give the upper and the lower bounds for the total charges. It is logical to consider the central estimation corresponding to $x = y = 0$. Calculations show that in some cases the central estimation is very close to the exact result.

The problem of two non-parallel disks was considered in (Rukhovets and Ufliand, 1971) using the Mehler-Fok transform. The following result was obtained for the total charge Q_1 in the assumption that

$$\mu_1/\sin(\alpha/2) \ll 1 \quad \text{and} \quad \mu_2/\sin(\alpha/2) \ll 1; \quad \mu_1 = R_1/d_1, \quad \mu_2 = R_2/d_2$$

$$Q_1 \approx \frac{2}{\pi} R_1 \left[V_1 - V_2 \frac{\mu_2}{\pi \sin(\alpha/2)} + V_1 \frac{\mu_1 \mu_2}{\pi^2 \sin^2(\alpha/2)} - V_2 \frac{\mu_1 \mu_2^2}{\pi^3 \sin^3(\alpha/2)} \right.$$

$$\left. + V_2 \frac{\mu_1^2 \mu_2 [2 + 3\sin^2(\alpha/2)]}{24\pi \sin^3(\alpha/2)} + V_2 \frac{\mu_2^3 [2 - 9\sin^2(\alpha/2)]}{24\pi \sin^3(\alpha/2)} \right]. \tag{3.2.27}$$

It is of interest to compare the results given by (25-26) with those by Rukhovets and Ufliand (27). Calculations were performed for the case $R_1 = R_2 = 1$, $V_1 = V_2 = 1$, $\alpha = \pi/4$, $d_1 = d_2 = l$. The value of $Q^* = \pi Q_1/2$ versus l is presented in Table 3.2.1.

Table 3.2.1. Comparison of our results with Rukhovets and Ufliand's

l	Upper bound for Q^*	Lower bound for Q^*	Central estimation	Rukhovets' result	Numerical solution
0.1	0.6521719	0.5000000	0.5114988	–	–
0.5	0.6858624	0.5000000	0.5612540	–	–
0.7	0.7013298	0.5000000	0.5871789	1.699512	–
1.0	0.7228720	0.5000000	0.6252061	1.023282	0.64925
1.5	0.7545003	0.6352847	0.6817026	0.8013288	0.70034
2.0	0.7811831	0.6964245	0.7271795	0.7774743	0.74027
3.0	0.8223547	0.7723489	0.7909440	0.8056570	0.79734
5.0	0.8731642	0.8500498	0.8595063	0.8626233	0.86149
7.0	0.9021068	0.8889649	0.8946465	0.8957692	0.89547
10.0	0.9273461	0.9203712	0.9235224	0.9239035	0.92383
15.0	0.9493124	0.9460188	0.9475592	0.9476710	0.94766

The numerical results were obtained by the method of iteration, with the accuracy 0.0001. As we can see, formula (27) gives good results for $l > 2.5$, the results sharply deviate from the admissible zone for $l < 2$. The central estimation gives reasonably good accuracy in the whole range $0 < l < \infty$.

The new approach allows a very simple treatment of complicated problems. The method is not limited to circular disks, it can be modified for other surfaces, for example, a system of arbitrarily located spherical caps can be treated in a similar manner.

Example 2. Consider the case of $n+1$ disks with their centers located at the plane $z=0$. The plane of the first disk is horizontal, its center being placed at the coordinates system origin, and its radius being a_0. This disk will be called central. The centers of the remaining n equal disks are located at the apices of a regular polygon, their planes being orthogonal to the line connecting the coordinates origin with the apex, the length of this line being l. Let the central disk be charged to a potential V_0, and the rest being kept at a potential V_1, and their radius being a_1. We need to write the set of approximate linear algebraic equations for the total charges on the disks.

Due to the symmetry of the system, the problem of finding the total charge at each disk can be reduced to a set of just two linear algebraic equations

$$Q_0 + \frac{2}{\pi} n Q_1 \sin^{-1}\left(\frac{a_0}{b_0}\right) = \frac{2}{\pi} V_0 a_0,$$

$$\frac{2}{\pi} Q_0 \sin^{-1}\left(\frac{a_1}{b_{01}}\right) + Q_1\left[1 + \frac{2}{\pi}\sum_{i=2}^{n} \sin^{-1}\left(\frac{a_1}{b_{11}}\right)\right] = \frac{2}{\pi} V_1 a_1,$$

where, from elementary geometrical considerations,

$$b_0 = \frac{1}{2}\{[(l + a_0)^2 + x^2]^{1/2} + [(l - a_0)^2 + x^2]^{1/2}\},$$

$$b_{01} = [(l - y)^2 + a_1^2]^{1/2},$$

$$b_{i1} = \frac{1}{2}[\sqrt{x_{i1}^2 + y_{i1}^2} + \sqrt{X_{i1}^2 + Y_{i1}^2}],$$

$$x_{i1} = 2h \sum_{k=1}^{i-2} \sin\left(\frac{2\pi k}{n}\right) + (h - a_1)\sin\left(\frac{2\pi(i-1)}{n}\right), \quad h = l\tan(\pi/n),$$

$$y_{i1} = 2h \sum_{k=1}^{i-2} \cos\left(\frac{2\pi k}{n}\right) + (h - a_1)\cos\left(\frac{2\pi(i-1)}{n}\right) + h + x,$$

$$X_{i1} = 2h \sum_{k=1}^{i-2} \sin\left(\frac{2\pi k}{n}\right) + (h + a_1)\sin\left(\frac{2\pi(i-1)}{n}\right),$$

$$Y_{i1} = 2h \sum_{k=1}^{i-2} \cos\left(\frac{2\pi k}{n}\right) + (h + a_1)\cos\left(\frac{2\pi(i-1)}{n}\right) + h + x,$$

for $i = 2, 3, ..., n$; where $-a_1 \le x \le a_1$ and $-a_0 \le y \le a_0$. The central estimation corresponds to the case $x = y = 0$. Note that the accuracy of the central estimation improves, as the ratio l/a_0 and l/a_1 increases, tending to the *exact* results when this ratio tends to infinity.

3.3. Capacity of flat laminae

A new method is proposed for the evaluation of the capacity of flat laminae of arbitrary shape. Specific approximate formulae are derived for evaluating the capacity of a polygon, a triangle, a rectangle, a rhombus, a circular sector and a circular segment. All the formulae are checked against the solutions known in the literature, and a good accuracy is confirmed.

Capacity is one of the most important electrostatic characteristics. Of all two-dimensional shapes, the exact formulae are known at the moment for a circle and for an ellipse only. There seems to be only one approximate formula for the capacity of a rectangle Howe (1920) which is not very good for the aspect ratios close to unity but gives better results for the rectangles with high aspect ratio. A universal formula was suggested by Solomon (1964a), who considered a mathematically equivalent contact problem of a flat punch on an elastic half-space. It is exact for a circle and reasonably accurate for a regular polygon but it fails even in the case of an ellipse of high eccentricity. A new method is suggested here giving simple yet accurate formulae for the capacity of flat laminae of arbitrary shape.

Consider a flat conducting lamina S whose boundary is given in the polar coordinates as

$$\rho = a(\phi). \tag{3.3.1}$$

The lamina is charged to a potential v. The governing integral equation takes the form

$$\iint_S \frac{\sigma(M)\,dS}{R(M,N)} = v(N). \tag{3.3.2}$$

Here σ is the charge density distribution, R is the distance between two points. By using (1.3.9), equation (2) can be written as

$$4\int_0^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_0^{2\pi} d\phi_0 \int_x^{a(\phi_0)} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}}\, \lambda\!\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right)\sigma(\rho_0,\phi_0) = v(\rho,\phi). \tag{3.3.3}$$

It is also noteworthy that the change of the order of integration which led to (3) is valid inside the circle $\rho \le \min a(\phi)$ only. Nevertheless, one can obtain from (3) *exact* solution for an ellipse and sufficiently accurate formulae for the capacity of flat laminae of arbitrary shape.

In the case of a conducting disk, the potential v is constant, and the capacity is defined as the ratio of the total charge Q and v. Let the charge density distribution be

$$\sigma = \frac{c\, a(\phi)}{\sqrt{a^2(\phi) - \rho^2}}. \tag{3.3.4}$$

where c is a constant which can be easily defined from the condition that the integral of σ over S should give the total charge Q.

$$\int_0^{2\pi} d\phi \int_0^{a(\phi)} \frac{c\, a(\phi)}{\sqrt{a^2(\phi) - \rho^2}}\, \rho\, d\rho = c \int_0^{2\pi} a^2(\phi)\, d\phi = 2Ac = Q, \tag{3.3.5}$$

where A is the area of S. It is noteworthy that the total charge does not depend on the location of the coordinate system origin. This location can be defined from the condition that the dipole moment of the charge distribution (4) about the origin be zero which leads to two equations

$$\int_0^{2\pi} a^3(\phi) \cos\phi\, d\phi = 0\,, \qquad \int_0^{2\pi} a^3(\phi) \sin\phi\, d\phi = 0. \tag{3.3.6}$$

The left hand sides of both equations are proportional to the x and y coordinates of the center of gravity which means that the origin of the system of polar coordinates should be located at the center of gravity of the lamina S. One gets immediately from (5) that

$$\sigma = \frac{Q\, a(\phi)}{2A\sqrt{a^2(\phi) - \rho^2}}. \tag{3.3.7}$$

For the case of a perfectly conducting lamina $v = \delta = const$. Now substituting (7) in (3), we can verify how close to a constant will be the potential produced by the charge distribution (7). Integration with respect to ρ_0 gives

$$v(\rho,\phi)=\frac{Q}{2A}\sum_{n=-\infty}^{\infty}\int_0^{\rho}\left(\frac{x}{\rho}\right)^{|n|}\frac{x\,\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_0^{2\pi}\mathrm{e}^{in(\phi-\phi_0)}F\left(1-\frac{|n|}{2},\frac{1}{2};1;1-\frac{x^2}{a^2(\phi_0)}\right)\mathrm{d}\phi_0. \tag{3.3.8}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the potential can be done separately for each harmonic. The zero*th* harmonic has the form

$$v_0=\frac{Q}{2A}\frac{\pi}{2}\int_0^{2\pi}a(\phi)\,\mathrm{d}\phi. \tag{3.3.9}$$

It is important to note that the second harmonic is equal zero for an arbitrary contour, and that all the odd harmonics will be zero if the expression for $a(\phi)$ does not contain odd harmonics. Here is the expression for the fourth harmonic

$$v_4=\frac{Q}{2A}\frac{8}{35}\rho^3\int_0^{2\pi}\frac{\mathrm{e}^{4i(\phi-\phi_0)}\,\mathrm{d}\phi_0}{a^2(\phi_0)}. \tag{3.3.10}$$

The investigation of further harmonics shows that their amplitude decreases.

Now consider in more detail the case of a square with the side $2l$. The equation of the boundary in this case is $a(\phi)=l/\cos\phi$ for $-\pi/4<\phi<\pi/4$, and the pattern is repeated outside this range. We can evaluate several non-zero harmonics:

$$v_0=\frac{Q}{2A}4\pi l\ln(1+\sqrt{2}),\qquad v_4=\frac{Q}{2A}\frac{32\rho^3\cos4\phi}{105l^2},$$

$$v_8=-\frac{Q}{2A}\frac{64}{3465}\rho\cos8\phi\left[\left(\frac{\rho}{l}\right)^2+\frac{12}{13}\left(\frac{\rho}{l}\right)^4+\frac{20}{13}\left(\frac{\rho}{l}\right)^6\right], \tag{3.3.11}$$

If we assume that the potential $\delta\approx v_0$ then the remaining harmonics may be called the solution error. Direct computations show that the error is less than 3% inside the circle $\rho\leq l$. The error is reasonably small outside the circle reaching 20% at the apex, and decreasing very rapidly with the distance from the apex. Taking into consideration that the error sign fluctuation will result in even smaller error in the total charge value, we may assume (9) being the relationship between the potential value and the total charge which can be rewritten in the form

$$\delta = \frac{Q}{g\sqrt{A}}. \tag{3.3.12}$$

where A is the area of the lamina, and g is a dimensionless coefficient depending on the lamina geometry only

$$g = \frac{2\sqrt{A}}{\pi^2 r_a}, \tag{3.3.13}$$

where r_a can be called *average radius* with respect to the center of gravity

$$r_a = \frac{1}{2\pi}\int_0^{2\pi} a(\phi)\,d\phi. \tag{3.3.14}$$

One can easily deduce that our coefficient g is related to the capacity C of a flat lamina by

$$g = \frac{C}{\sqrt{A}}. \tag{3.3.15}$$

The problem now is to find the value of g for various shapes. One can easily compute the coefficient g for the square from the first equation (11) as

$$g = \frac{1}{\pi\, ln(1+\sqrt{2})} = 0.3611$$

which is very close to the value 0.3607 given by Maxwell for the capacitance of the square. Of course, closeness to the result by Maxwell does not mean that our result is so accurate. The value of g which seems to be accurate was obtained by Noble (1960), and is 0.367, so that our result is in error by 1.6% which is not bad. Now it seems logical to assume that formulae (12–14) are valid for an arbitrary lamina, and we shall verify how good they really are for each specific case in the next Section.

We have found in the literature only one general formula of the type (12) suggested by Solomon (1964a). His result expressed through the coefficient g reads

$$g = \frac{2^{9/8} I_0^{1/8}}{\pi^{11/8} A^{1/4}}, \tag{3.3.16}$$

where I_0 stands for the polar moment of inertia. One can easily verify that formula (16) is exact for a circle, so one should expect it to be sufficiently accurate for domains with the aspect ratio not far away from unity, but the error might be quite significant for oblong domains. For example, in the case of an ellipse with semi-axes a and b formula (16) gives

$$g = \frac{2^{7/8}}{\pi^{3/2}}\left(\frac{a}{b}+\frac{b}{a}\right)^{1/8}.$$

Our formulae (12–14) in the case of an ellipse are exact. Several specific applications are considered below.

Polygon. Consider a polygon with n sides, with the only limitation that the function a describing its boundary be continuous and single-valued. The origin of the coordinate system is located at the center of gravity, as before. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the k-th side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered k+1. It is clear that the value of index equal n+1 is understood as 1. Denote b_k the distance from the center of gravity to the k-th apex. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. Then formulae (13) and (14) yield the following expression for the coefficient g

$$g = \frac{2\sqrt{A}}{\pi\sum_{k=1}^{n}\frac{A_k}{a_k}\ln\frac{b_k+b_{k+1}+a_k}{b_k+b_{k+1}-a_k}}. \tag{3.3.17}$$

In the case of a regular polygon formula (17) simplify to

$$g = \frac{4\sqrt{\tan(\pi/n)}}{\pi\sqrt{n}\ \ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}}\ . \tag{3.3.18}$$

Consider several particular values of n. For an equilateral triangle (n=3) formula (18) gives g = 0.3673. The value of g, which seems to be accurate, can be computed from (Solomon, 1964b), and is equal 0.3829, so that our result is in error by 4.1%. As we have seen earlier, the error of (18) for a square is 1.6%. Since formula (18) in the limiting case $n\to\infty$ gives the exact result for a circle $g = 2/\pi^{3/2} = .35917$, we should expect that the error of (18) will decrease with n. For a regular pentagon g = 0.3599. We did not find in the literature anything to compare with this result. It seems quite clear that the maximum possible error indeed decreases with n. It is noteworthy that the value of g does not change significantly in the whole range $3 \le n < \infty$.

Triangle. In the case of a triangle with the sides a_1, a_2 and a_3, formula (17) simplifies as follows:

$$g = \frac{6}{\pi\sqrt{A}}\left[\frac{1}{a_1}\ln\frac{b_1+b_2+a_1}{b_1+b_2-a_1} + \frac{1}{a_2}\ln\frac{b_2+b_3+a_2}{b_2+b_3-a_2} + \frac{1}{a_3}\ln\frac{b_3+b_1+a_3}{b_3+b_1-a_3}\right]^{-1}. \tag{3.3.19}$$

The parameters in (19) can be defined by the well known formulae from geometry:

$$A = [\,p(p-a_1)(p-a_2)(p-a_3)\,]^{1/2}, \quad p = (a_1+a_2+a_3)/2,$$

$$b_1 = \frac{1}{3}[\,2(a_1^2+a_3^2)-a_2^2\,]^{1/2}, \qquad b_2 = \frac{1}{3}[\,2(a_2^2+a_1^2)-a_3^2\,]^{1/2},$$

$$b_3 = \frac{1}{3}[\,2(a_3^2+a_2^2)-a_1^2\,]^{1/2}.$$

We are unaware of any report treating a triangle of general type but certain particular cases have been considered, so we can compare the results. When $a_1 = a_2 = l$, and the angle between these two sides is equal α, formula for the coefficient g can be rewritten in the form

$$g = \frac{6}{\pi}\sqrt{\tan(\alpha/2)}\left[2\sin(\frac{\alpha}{2})\ln(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}) + \ln\tan(\frac{\pi}{4}+\frac{\gamma}{2})\right]^{-1}, \tag{3.3.20}$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

In the case of $\alpha = \pi/2$, Okon and Harrington (1970) obtained $g = 0.3867$ as the most probable result. Our result is $g = 0.374$ which is within 3.3% from the numerical one.

Rectangle. Consider a rectangular lamina, a and b being its semiaxes. Introduce the aspect ratio $\varepsilon = a/b$. Formula (17) in this case reduces to

$$g = \frac{2}{\pi[\sqrt{\varepsilon}\sinh^{-1}(1/\varepsilon) + (1/\sqrt{\varepsilon})\sinh^{-1}\varepsilon]}. \tag{3.3.21}$$

Howe (1920) suggested an approximate formula for the capacitance of a rectangle which in terms of the coefficient g reads

$$g = \frac{1}{2\sqrt{\varepsilon}\left[\frac{1}{\varepsilon}\sinh^{-1}\varepsilon + \sinh^{-1}(\frac{1}{\varepsilon}) + \frac{\varepsilon}{3} + \frac{1}{3\varepsilon^2} - \frac{(\varepsilon^2+1)^{3/2}}{3\varepsilon^2}\right]}, \quad (3.3.22)$$

The result due to Solomon (16) in this case takes the form

$$g = \frac{2^{(9/8)}}{\pi^{11/8}}\left[\frac{\varepsilon}{12} + \frac{1}{12\varepsilon}\right]^{1/8}. \quad (3.3.23)$$

We have found in the literature some numerical results which seem to be more or less accurate. Noble (1960) investigated a problem of electric charge distribution on a rectangular lamina, and Borodachev and Galin (1974) have considered an equivalent problem of a narrow rectangular punch on an elastic half-space. Their data, expressed in terms of the coefficient g, is presented below and compared with our result (21) and those due to Howe (22) and Solomon (23). The following relationship was used between Borodachev-Galin's coefficient γ and our g: $\gamma=1/2\pi g\sqrt{\varepsilon}$.

ε =	0.020	0.050	0.100	0.125	0.150	0.200	0.250	0.500	1.000
Borodachev and Galin	0.7375	0.5661	0.4819	–	0.4458	0.4259	–	–	–
Noble	–	–	–	0.4543	–	–	0.4047	0.3762	0.3670
Formula (21)	0.8031	0.6072	0.5037	0.4771	0.4576	0.4306	0.4128	0.3742	0.3612
Howe (22)	0.6916	0.5317	0.4481	0.4268	0.4112	0.3899	0.3759	0.3462	0.3363
Solomon (23)	0.5402	0.4819	0.4423	0.4304	0.4211	0.4071	0.3969	0.3715	0.3613
Discrepancy %									
Formula (21)	-8.9	-7.3	-4.5	-5.0	-2.6	-1.1	-2.0	0.5	1.6
Howe (22)	6.2	6.1	7.0	6.1	7.8	8.5	7.1	8.0	8.4
Solomon (23)	26.7	14.9	8.2	5.3	5.5	4.4	1.9	1.3	1.6

Several useful conclusions can be drawn from the data presented. It seems logical to assume that the error of an approximate formula should change monotonously (or to have only one extremum) with respect to a certain parameter. The fact that the discrepancy due to each formula jumps when moving from the data due to Noble to those by Borodachev and Galin, indicates that the results of at least one author are not exact. Our own computations favor the results of Noble. Here are some of our numerical results: $\varepsilon=0.5$, $g=0.3763$; $\varepsilon=1/8$, $g=0.4543$; $\varepsilon=0.1$, $g=0.4752$; $\varepsilon=1/15$, $g=0.5200$; $\varepsilon=1/30$, $g=0.6207$; $\varepsilon=1/40$, $g=0.6729$; $\varepsilon=1/50$, $g=0.7192$. Our formula seems to perform better then the other two in a sufficiently wide range of the aspect ratio. As it was expected formula due to Solomon performs well when the aspect ratio is not far away from unity only. There seems to be little change in the error of Howe's formula (22). If this is really so, then its accuracy can be improved dramatically just by multiplication by a constant factor, say, 1.07.

Rhombus. Let α be the angle at one of the rhombus apexes. Formula (17) in this case yields

$$g = \frac{2}{\pi\sqrt{\sin\alpha}\,\ln\dfrac{\cos(\alpha/2)+\sin(\alpha/2)+1}{\cos(\alpha/2)+\sin(\alpha/2)-1}}. \tag{3.3.24}$$

The same formula in terms of the rhombus semiaxes a and b and the aspect ratio $\varepsilon = a/b$ has the form

$$g = \frac{\sqrt{2(\varepsilon+(1/\varepsilon))}}{\pi\ln\dfrac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}}. \tag{3.3.25}$$

The capacity of a diamond was computed by Okon and Harrington (1970). Their result, expressed in terms of the coefficient g, for a diamond with the aspect ratio a:b = 0.7:1.65 is g = 0.3855. Formula (25) gives g = 0.3744 which is within 3% from the result of Okon and Harrington. They also considered a rhombus with the aspect ratio 1:2. Their result g = 0.3705 practically coincides with ours g = 0.3698. Slightly different numerical results are given by De Smedt (1979). We compare his data expressed in terms of the coefficient g with those computed due to (25)

ε=	0.100	0.200	0.333	0.500	0.750	1.000
De Smedt (1979)	0.521	0.442	0.402	0.381	0.369	0.366
Formula (25)	0.462	0.409	0.383	0.370	0.363	0.361
Discrepancy %	11.4	7.6	4.8	2.8	1.6	1.3

Our formula (25) seems to perform quite satisfactory in a wide range of the aspect ratio ε.

Circular segment. Let the radius r and the angle 2α be the segment parameters. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}. \tag{3.3.26}$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r\,[-k\cos\phi + \sqrt{1-k^2\sin^2\phi}, \quad \text{for} \quad 0\le\phi\le\pi-\gamma \quad \text{or} \quad \pi+\gamma\le\phi<2\pi;$$

$$a(\phi) = r\,\frac{k-\cos\alpha}{\cos(\pi-\phi)}, \quad \text{for} \quad \pi-\gamma\le\phi\le\pi+\gamma. \tag{3.3.27}$$

Substitution of (27) into (13)–(14) yields

$$g = \frac{2\sqrt{\alpha - \sin\alpha\cos\alpha}}{\pi\{2E(k) - E(\gamma,k) - k\sin\gamma + (k - \cos\alpha\ln\tan[(\pi+2\gamma)/4]\}}. \tag{3.3.28}$$

where $\gamma = \tan^{-1}[\sin\alpha/(k - \cos\alpha)]$. We have found only one numerical example to verify the accuracy of (28): Okon and Harrington (1970) have computed the capacity of a semicircle. Their result expressed through the coefficient g is 0.3724. Formula (28) gives 0.3714 with the discrepancy of 0.3%.

Lune. Two equal circular segments of radius r and angle 2α joined along their chords give us a lamina of lune shape. The capacity of such lamina was considered by Lebedev *et. al.*(1986). They have obtained an asymptotic formula for narrow lune which reads

$$g = \frac{2\sin\alpha}{L\sqrt{2\alpha - \sin 2\alpha}}\left[1 - \frac{2}{L} + \frac{8}{L^2}\left(1 - \frac{\pi^2}{24}\right) + \frac{130}{3}\left(1 - \frac{3\pi^2}{65}\right)\frac{e^{-L}}{L^2}\right]. \tag{3.3.29}$$

Here

$$L = \frac{\pi K(\cos\alpha)}{K(\sin\alpha)},$$

and $K()$ stands for the complete elliptic integral of the first kind. Our method yields

$$g = \frac{\sqrt{2\alpha - \sin 2\alpha}}{\pi[E(\cos\alpha) - \cos\alpha]}. \tag{3.3.30}$$

Here $E()$ denotes a complete elliptic integral of the second kind. It is of interest to compare formulae (29) and (30) with an accurate numerical solution obtained by the method described in section 7.3. The results are given below

α (degrees)	5	7	10	15	20	30	45	60
g (29)	0.6277	0.5753	0.5300	0.4923	0.4763	0.4795	0.5624	0.8287
g (30)	0.5749	0.5265	0.4831	0.4426	0.4191	0.3927	0.3737	0.3647
g numerical	0.6283	0.5656	0.5135	0.4631	0.4340	0.4044	0.3784	0.3662
error of (30) (%)	8.5	6.9	5.9	4.4	3.4	2.9	1.2	0.4

Our formula (30) is sufficiently accurate for $\alpha > 15$ degrees. Lebedev's formula (29) is good only for very small $\alpha < 15$ degrees (although the authors claim it to be sufficiently accurate up to $\alpha = 30$ degrees). It deviates from reasonable behavior for $\alpha > 20$ degrees (g increases instead of decreasing). In view of these properties of (29), we can suggest a much more simple asymptotic formula, namely,

$$g = \frac{\sqrt{3}}{2\sqrt{\alpha}\,\ln(4/\alpha)}\left[1 - \frac{1}{\ln(4/\alpha)} + \frac{2}{\ln^2(4/\alpha)}\left(1 - \frac{\pi^2}{24}\right)\right]. \qquad (3.3.31)$$

Besides being more simple than (29) formula (31) is also more accurate. Some numerical results for a 'shamrock'-shaped lamina can be found in section 7.3 where a mathematically equivalent contact problem is considered.

Circular sector. Repetition of the procedure, described earlier, leads to the following result for a circular sector with the angle 2α:

$$g = \frac{2\sqrt{\alpha}}{\pi\{E(\gamma, k) - k\sin\gamma + k\sin\alpha\,\ln[\cot(\alpha/2)\cot((\gamma-\alpha)/2)]\}}. \qquad (3.3.32)$$

Here, $k=2\sin\alpha/(3\alpha)$, and $\gamma=\tan^{-1}(\sin\alpha/(\cos\alpha-k))$. Okon and Harrington (1970) in the case of a quadrant obtained $g=0.3668$. Formula (32) for $\alpha=\pi/4$ gives $g=0.3639$, with the discrepancy of 0.8%.

One can enquire whether there exists any contour, other than an ellipse, for which expression of the type (7) would be an exact solution to the integral equation (29). Expression (8) can provide the sufficient conditions:

$$\int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(1 - \frac{|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) d\phi_0 \qquad (3.3.33)$$

should be equal zero for $n\neq 0$. Integral (33) will vanish for all odd n if $a(\phi)$ contains even harmonics only. In the case of even n, the hypergeometric function in (33) represents a finite polynomial in $x/a(\phi)$ of degree not greater than $n-2$ which means that integral (33) vanishes if $[a(\phi)]^{-2}$ contains harmonics not higher than the second, which corresponds to an ellipse. The question whether these conditions will be necessary requires an additional investigation.

Solomon's formula (16) can be considered as a particular case of a more general one, namely

$$g = \frac{2}{\pi\sqrt{A}}\left[\frac{1}{2\pi}\int_0^{2\pi} (a(\phi))^m d\phi\right]^{1/2m}\left[\frac{1}{2\pi}\int_0^{2\pi} (a(\phi))^n d\phi\right]^{1/2n} \qquad (3.3.34)$$

for $m=2$ and $n=4$. One may ask now whether this choice of the parameters m and n is in any sense optimal. Direct computations show that this is not the case. Different shapes require different values for m and n. Formulae of type (34) are of empirical type, and have very little physical background.

3.4. Magnetic polarizability of small apertures

Analysis of magnetic polarizability of small apertures of arbitrary shape is presented here. A general formula is derived for the coefficients of magnetic polarizability of small apertures. Specific formulae are obtained for the apertures of various shape. All the formulae are checked against the solutions known in the literature, and their accuracy is confirmed. The material follows (Fabrikant, 1987b).

Many years ago Bethe (1944) reduced the problem of diffraction by small apertures to an evaluation of the coefficient of electric polarizability and the tensor of magnetic polarizability. At the moment, closed-form expressions for these quantities are known for an elliptic aperture in a planar screen only. All non-elliptic shapes have been treated either experimentally (Cohn 1951) or numerically (Okon and Harrington 1981, de Smedt 1979, De Meulenaere and Van Bladel 1977), the variational approach was used by Fikhmanas and Fridberg (1973). Their results will be used for verification of the accuracy of the formulae to be derived here.

It is well known (Bethe, 1944) that the problem of diffraction by small apertures can be reduced to the solution of the following integral equation

$$w(N) = \int\int_S \frac{\sigma(M)}{R(M,N)}\,\mathrm{d}S, \tag{3.4.1}$$

where S is a two-dimensional domain of the aperture, $R(M,N)$ stands for the distance between the points M and N, w is a known function, and σ stands for the charge density (unknown function), Let the boundary of the aperture S in a planar screen be given in the polar coordinates as

$$\rho = a(\phi),$$

where the function $a(\phi)$ is bounded and single-valued. We use again the integral representation for the reciprocal distance established in Chapter 1

$$\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\mathrm{d}x}{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}, \tag{3.4.2}$$

where

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}. \tag{3.4.3}$$

Substitution of (2) into (1) gives, after changing the order of integration

$$w(\rho, \phi) = \frac{2}{\pi} \int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} d\phi_0 \int_x^{a(\phi_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0)}{(\rho_0{}^2 - x^2)^{1/2}} \sigma(\rho_0, \phi_0)\, \rho_0 d\rho_0. \tag{3.4.4}$$

It is noteworthy that the change of the order of integration which led to (4) is valid inside the circle $\rho \le \min\{a(\phi)\}$ only. Nevertheless, one can obtain from (4) the *exact* solution for an ellipse and sufficiently accurate formulae for various specific apertures as it will be demonstrated further.

For the case of magnetic polarizability, it is sufficient to consider equation (1), with the function w taking the form

$$w = \alpha_x y - \alpha_y x, \tag{3.4.5}$$

where α_x and α_y are constants. It is quite clear that in the case of a uniaxial excitation one of these constants can be put equal zero.

Let the charge distribution at the aperture be

$$\sigma(\rho, \phi) = \frac{a(\phi)\, \rho(p_1 \cos\phi + p_2 \sin\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}, \tag{3.4.6}$$

where p_1 and p_2 are yet unknown constants. The main reason for this choice is the requirement that (6) be exact for an ellipse. Make use of the condition that the integral of σ over S should be equal zero. Since p_1 and p_2 are independent, this leads to two equations

$$\int_0^{2\pi} (a(\phi))^3 \cos\phi\, d\phi = 0\,, \qquad \int_0^{2\pi} (a(\phi))^3 \sin\phi\, d\phi = 0. \tag{3.4.7}$$

One can note that the left-hand side of each equation (7) is proportional to the x or y coordinates of the center of gravity. This means that the origin of the system of coordinates should be located at the center of gravity of the aperture. The axis orientation will be discussed later.

The relationships between the dipole moments and the parameters p_1 and p_2 can be established from the conditions

$$M_x = \int\!\!\int_S \sigma y\, dS\,, \qquad M_y = -\int\!\!\int_S \sigma x\, dS,$$

which leads to

$$M_x = \frac{8}{3}(p_1 I_{xy} + p_2 I_x)\,, \qquad M_y = -\frac{8}{3}(p_1 I_y + p_2 I_{xy}), \tag{3.4.8}$$

where I_x, I_y and I_{xy} are the well known quantities of the moments of inertia and the product of inertia respectively.

$$I_x = \int\!\!\int_S y^2 \mathrm{d}S\,, \quad I_y = \int\!\!\int_S x^2 \mathrm{d}S\,, \quad I_{xy} = \int\!\!\int_S xy\,\mathrm{d}S.$$

Now it is necessary to relate p_1 and p_2 to the parameters α_x and α_y. This can be done by substitution of (6) into (4) which yields after integration with respect to ρ_0

$$w(\rho,\phi) = \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x^2\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \mathrm{e}^{in(\phi-\phi_0)}$$

$$\times F\left(\frac{3-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right)(p_1 \cos\phi_0 + p_2 \sin\phi_0)\,\mathrm{d}\phi_0. \tag{3.4.9}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the function w can be done separately for each harmonic. Note that the zero*th* and all the even harmonics of w will be zero if $a(\phi)$ contains only the even harmonics. The first harmonic will take the form

$$w_1(\rho,\phi) = \frac{\pi}{2}\rho \int_0^{2\pi} \cos(\phi - \phi_0)\,(p_1 \cos\phi_0 + p_2 \sin\phi_0\,)\,a(\phi_0)\,\mathrm{d}\phi_0,$$

which can be simplified as

$$w_1(\rho,\phi) = \frac{\pi}{2}\rho\,[(p_1 J_y + p_2 J_{xy})\cos\phi + (p_1 J_{xy} + p_2 J_x)\sin\phi]. \tag{3.4.10}$$

Here the following quantities were introduced

$$J_x = \int_0^{2\pi} a(\phi)\sin^2\phi\,\mathrm{d}\phi\,, \quad J_y = \int_0^{2\pi} a(\phi)\cos^2\phi\,\mathrm{d}\phi\,, \quad J_{xy} = \int_0^{2\pi} a(\phi)\sin\phi\cos\phi\,\mathrm{d}\phi. \tag{3.4.11}$$

These quantities do not seem to have been used before in engineering practice so they do not have an accepted name. Since their tensor properties are similar to those of the moments of inertia, we shall call J_x and J_y *the linear moments of a two-dimensional domain about the axes Ox and Oy* respectively, J_{xy} will be called *the linear product of a two-dimensional domain about the axes Ox and Oy*

It is important to note that the third harmonic is equal zero for an arbitrary contour. Here is the expression for the fifth harmonic

$$w_5(\rho, \phi) = \frac{128}{315}\rho^4 \int_0^{2\pi} \frac{\cos 5(\phi - \phi_0)}{a^2(\phi_0)} (p_1 \cos\phi_0 + p_2 \sin\phi_0)\, d\phi_0,$$

which can be modified as

$$w_5(\rho, \phi) = \frac{64}{315}\rho^4 \{[(A_{c6}+A_{c4})p_1 + (A_{s6} - A_{s4})p_2] \cos 5\phi$$

$$+ [(A_{s6}+A_{s4})p_1 + (A_{c4} - A_{c6})p_2] \sin 5\phi\}. \qquad (3.4.12)$$

Here, the following geometrical characteristics of the aperture domain were introduced

$$A_{c4} = \int_0^{2\pi} \frac{\cos 4\phi\, d\phi}{(a(\phi))^2}, \qquad A_{c6} = \int_0^{2\pi} \frac{\cos 6\phi\, d\phi}{(a(\phi))^2},$$

$$A_{s4} = \int_0^{2\pi} \frac{\sin 4\phi\, d\phi}{(a(\phi))^2}, \qquad A_{s6} = \int_0^{2\pi} \frac{\sin 6\phi\, d\phi}{(a(\phi))^2}.$$

Investigation of further harmonics shows that their amplitude decreases. Since the amplitude of w_5 is significantly less than that of w_1, it seems natural to assume $w \approx w_1$, and the remaining harmonics may be called the solution error. Taking into consideration that the error sign fluctuation will result in even smaller error in the integral characteristics sought, a direct comparison of (5) and (10) leads to

$$\alpha_x = \frac{\pi}{2}(p_1 J_{xy} + p_2 J_x), \qquad \alpha_y = -\frac{\pi}{2}(p_1 J_y + p_2 J_{xy}). \qquad (3.4.13)$$

The inversion of (13) gives

$$p_1 = -\frac{2}{\pi}\,\frac{J_{xy}\alpha_x + J_x\alpha_y}{J_x J_y - J_{xy}^2}, \qquad p_2 = \frac{2}{\pi}\,\frac{J_y\alpha_x + J_{xy}\alpha_y}{J_x J_y - J_{xy}^2}. \qquad (3.4.14)$$

Substitution of (14) in (8) finally gives the required relationship

$$M_x = \frac{16}{3\pi}(m_{11}\alpha_x + m_{12}\alpha_y), \qquad M_y = \frac{16}{3\pi}(m_{21}\alpha_x + m_{22}\alpha_y), \tag{3.4.15}$$

where

$$m_{11} = \frac{J_y I_x - J_{xy} I_{xy}}{J_x J_y - J_{xy}^2}, \qquad m_{12} = \frac{J_{xy} I_x - J_x I_{xy}}{J_x J_y - J_{xy}^2},$$

$$m_{21} = \frac{J_{xy} I_y - J_y I_{xy}}{J_x J_y - J_{xy}^2}, \qquad m_{22} = \frac{J_x I_y - J_{xy} I_{xy}}{J_x J_y - J_{xy}^2}.$$

It is clear that all these results can be rewritten in a matrix or a tensor form. One can verify that formulae (15) are invariant with respect to an arbitrary rotation of the axes. The same property holds for $m_{11} + m_{22}$ and $m_{12} - m_{21}$. Strictly speaking, according to the reciprocal theorem, m_{12} should be equal m_{21}, so that formulae (15) generally do not satisfy this theorem. But we may state that this theorem is satisfied 'approximately'. We mean by this the following property which has been verified by several direct computations, namely, $|m_{12} - m_{21}|/m_{11} \ll 1$ and $|m_{12} - m_{21}|/m_{22} \ll 1$. This theorem will be satisfied exactly for any domain which has at least one axis of symmetry because in this case $m_{12} = m_{21} = 0$ provided that the coordinate axes coincide with the central principal axes of the domain of aperture. Since we have no numerical data for non-symmetrical domains which could be used to verify the accuracy of (15), we shall consider further only the case when the aperture has an axis of symmetry. In this case formulae (8), (13) and (15) simplify significantly

$$M_x = \frac{8}{3} I_x p_2, \qquad M_y = -\frac{8}{3} I_y p_1, \tag{3.4.16}$$

$$\alpha_x = \frac{\pi}{2} J_x p_2, \qquad \alpha_y = -\frac{\pi}{2} J_y p_1, \tag{3.4.17}$$

$$M_x = \frac{16}{3\pi}\frac{I_x}{J_x}\alpha_x, \qquad M_y = \frac{16}{3\pi}\frac{I_y}{J_y}\alpha_y. \tag{3.4.18}$$

Now, we can rewrite the expression for the charge distribution (6) in terms of the moments M_x and M_y in the form

$$\sigma = \frac{a(\phi)}{[a^2(\phi) - \rho^2]^{1/2}}\left[\frac{3}{4}\left(\frac{M_x y}{I_x} - \frac{M_y x}{I_y}\right)\right]. \tag{3.4.19}$$

An expression equivalent to (19) can be written in terms of the parameters α_x and α_y

$$\sigma = \frac{2a(\phi)}{\pi[a^2(\phi) - \rho^2]^{1/2}} \left[\frac{\alpha_x y}{J_x} - \frac{\alpha_y x}{J_y} \right]. \tag{3.4.20}$$

Expressions (19) and (20) are *exact* for an ellipse. We expect them to be reasonably accurate in the neighborhood of the coordinate origin for an arbitrary aperture with at least one axis of symmetry, while the error might become quite significant close to the boundary of the domain S.

Let us rewrite formulae (18) in the form

$$M_x = \frac{A^{3/2}}{2\pi} \nu_x \alpha_x, \qquad M_y = \frac{A^{3/2}}{2\pi} \nu_y \alpha_y, \tag{3.4.21}$$

where A is the aperture area, and

$$\nu_x = \frac{32 I_x}{3 A^{3/2} J_x}, \qquad \nu_y = \frac{32 I_y}{3 A^{3/2} J_y}. \tag{3.4.22}$$

We introduced the coefficients ν_x and ν_y for two reasons: since they are dimensionless they characterize the shape of S and do not depend on its size; both ν_x and ν_y are equal to the corresponding coefficients of magnetic polarizability which will simplify the comparison of our results with the numerical data available. The remaining part of the section will be devoted to the evaluation of the coefficients ν_x and ν_y for various aperture shapes.

Several specific aperture shapes are considered here. Each configuration is related to its central principal axes and assumed to have at least one axis of symmetry coinciding with the axis Ox.

Polygon. Consider a polygon with n sides. The function $a(\phi)$ describing its boundary is bounded and single-valued. The origin of the coordinate system is located at the center of gravity, as before. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered $k+1$. It is clear that the value of index equal $n+1$ is understood as 1. Denote b_k the distance from the center of gravity to the kth apex; ψ_k stands for the angle between the axis Ox and the perpendicular to the side a_k. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. The following expressions can be obtained for the moments of inertia

$$I_x = \sum_{k=1}^{n} -m_k \cos 2\psi_k + g_k \sin 2\psi_k + 2h_k \cos^2\psi_k \, ,$$

$$I_y = \sum_{k=1}^{n} m_k \cos 2\psi_k - g_k \sin 2\psi_k + 2h_k \sin^2\psi_k ,$$

$$I_{xy} = \sum_{k=1}^{n} (m_k - h_k) \sin 2\psi_k + g_k \cos 2\psi_k , \tag{3.4.23}$$

where

$$m_k = \frac{2A_k^3}{a_k^2} , \quad g_k = A_k^2 \frac{b_{k+1}^2 - b_k^2}{2a_k^2} , \quad h_k = \frac{A_k[3(b_{k+1}^2 + b_k^2) - a_k^2]}{24} . \tag{3.4.24}$$

Formulae (23–24) are valid for an arbitrary polygon, not necessarily having an axis of symmetry.

The linear moments can be computed in the form

$$J_x = \sum_{k=1}^{n} -q_k \cos 2\psi_k + s_k \sin 2\psi_k + 2t_k \cos^2\psi_k \, ,$$

$$J_y = \sum_{k=1}^{n} q_k \cos 2\psi_k - s_k \sin 2\psi_k + 2t_k \sin^2\psi_k \, , \tag{3.4.25}$$

$$J_{xy} = \sum_{k=1}^{n} (q_k - t_k) \sin 2\psi_k + s_k \cos 2\psi_k ,$$

where

$$q_k = \frac{A_k}{a_k^2}\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right)[a_k^2 + (b_k - b_{k+1})^2] \, , \quad s_k = \frac{4A_k^2}{a_k^2}\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right),$$

$$t_k = \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k} . \tag{3.4.26}$$

Substitution of (23–26) into (22) gives the coefficients ν_x and ν_y for an arbitrary polygon. In the case of a regular polygon $a_k = a$, $b_k = b = a/[2\sin(\pi/n)]$, $\psi_k = 2\pi(k-1)/n$, $A_k = [a^2\cot(\pi/n)]/4 = [b^2\sin(2\pi/n)]/2$, $A = nA_k$, and formulae (23–26) simplify to

$$I_x = I_y = \frac{na^4}{64}\cot\frac{\pi}{n}\left[\cot^2\frac{\pi}{\mathrm{n}} + \frac{1}{3}\right] = \frac{nb^4}{24}\sin\frac{2\pi}{n}\left[2+\cos\frac{2\pi}{n}\right], \tag{3.4.27}$$

$$J_x = J_y = \frac{1}{4}na\cot\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)} = \frac{1}{2}nb\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}. \tag{3.4.28}$$

Substituting (27) and (28) in (22) leads to

$$\nu_x = \nu_y = \frac{16(2+\cos\frac{2\pi}{n})}{9\left(n^3\sin\frac{\pi}{n}\cos^3(\frac{\pi}{\mathrm{n}})\right)^{1/2}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}}. \tag{3.4.29}$$

Consider several particular values of n. For an equilateral triangle (n=3) formula (29) gives $\nu_x = \nu_y = 3^{1/4}16/[27\ln(2+\sqrt{3})] = 0.5922$. We did not find any numerical data to compare with this result. In the case of a square $n = 4$, and $\nu_x = \nu_y = 4/[9\ln(1+\sqrt{2})] = 0.5043$ which is inside the interval from 0.4973 to 0.5162 given by Okon and Harrington (1981) and within 3% from the result of de Smedt 0.5193. Since formula (29) in the limiting case $n\to\infty$ gives the exact result for a circle $\nu_x = \nu_y = 8/(3\pi^{3/2}) = 0.4789$, we should expect that the error of (29) will decrease with n. The value of the coefficients for a regular hexagon is $\nu_x = \nu_y = 40\sqrt{2}/(3^{1/4}81\ln 3) = 0.4830$ which differs by 1.4% from the result 0.49 due to Okon and Harrington (1981), and it is quite clear that the maximum possible error indeed decreases with n. It is noteworthy that the coefficients of magnetic polarizability do not change significantly in the whole range $3 \le n < \infty$.

Isosceles triangle. In the case of a triangle with the sides $a_1 = a_2 = l$ and the angle between them equal to α formulae (22–26) give

$$I_x = \frac{1}{12}l^4\sin\alpha\sin^2(\alpha/2), \qquad I_y = \frac{1}{36}l^4\sin\alpha\cos^2(\alpha/2),$$

$$J_x = \frac{2}{3}l\cos\frac{\alpha}{2}\Big[\sin\alpha + \sin(\alpha+\gamma) - 2\sin\gamma$$

$$+ 2\sin^3\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right) + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\Bigg]\Bigg],$$

$$J_y = \frac{2}{3}\,l\cos\frac{\alpha}{2}\left[-\sin\alpha - \sin(\alpha+\gamma) + 2\sin\gamma + \sin\alpha\cos\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right)\right],$$

with the result for the coefficients

$$\nu_x = 8(\tan(\alpha/2))^{3/2}\Bigg\{3\Bigg[\sin\alpha + \sin(\alpha+\gamma) - 2\sin\gamma$$

$$+ 2\sin^3\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right) + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\Bigg]\Bigg\}^{-1},$$

$$\nu_y = 8\sqrt{\cot(\alpha/2)}\;\Bigg\{9\Bigg[-\sin\alpha - \sin(\alpha+\gamma) + 2\sin\gamma$$

$$+ \sin\alpha\cos\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right)\Bigg]\Bigg\}^{-1}, \qquad (3.4.30)$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

The isosceles right triangle was considered by Okon and Harrington (1981) who gave the interval between 0.9829 and 1.021 for only one coefficient which in our notation is ν_x. Our result for ν_x is 0.9255 which differs less than 10% from theirs. The second formula (30) gives $\nu_y = 0.3995$, and there is nothing in the literature to compare with this result.

Rectangle. Consider a rectangular aperture, a_1 and a_2 being its semiaxes. Introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (23–26) in this case reduce to

$$I_x = \tfrac{4}{3}a_1a_2^3, \qquad I_y = \tfrac{4}{3}a_1^3a_2,$$

$$J_x = 4a_1\sinh^{-1}\varepsilon, \qquad J_y = 4a_2\sinh^{-1}(1/\varepsilon),$$

and formulae (22) yield

$$\nu_x = \frac{4\varepsilon^{3/2}}{9\sinh^{-1}\varepsilon}, \qquad \nu_y = \frac{4\varepsilon^{-3/2}}{9\sinh^{-1}(1/\varepsilon)}. \tag{3.4.31}$$

The coefficients of magnetic polarizability were computed by de Smedt (1979) for a rectangle with different aspect ratio ε. Here, we present his results along with those given by (31).

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x=	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (31) ν_x=	0.1408	0.2001	0.2612	0.3265	0.4165	0.4341	0.5043
de Smedt ν_y=	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (31) ν_y=	4.6876	2.1488	1.2701	0.8708	0.6228	0.5929	0.5043
Discrepancy in ν_x (%)	-9.4	-6.4	-3.2	-0.5	1.8	2.2	2.9
Discrepancy in ν_y (%)	-14.1	-6.1	-0.8	2.1	3.1	3.3	2.9

Our formula (31) seems to perform satisfactorily in a sufficiently wide range of aspect ratio. The approximate expression for the charge distribution at the aperture, according to (19), takes the form

$$\sigma = \frac{a(\phi)}{8a_1a_2[\,a^2(\phi) - \rho^2\,]^{1/2}}\left[\frac{9}{4}\left(\frac{M_x y}{a_2^2} - \frac{M_y x}{a_1^2}\right)\right]. \tag{3.4.32}$$

The results due to (32) can be compared with the numerical data received in personal communication from de Smedt. In order to make the comparison possible, one should put in (32) M_x=0, replace M_y by (21), with the result

$$\sigma = \frac{9\sqrt{\varepsilon}\,a(\phi)\,\nu_y\,x}{4a_1[\,a^2(\phi) - \rho^2\,]^{1/2}}. \tag{3.4.33}$$

Computations due to (33) were made for $\varepsilon = 0.5$ along the axis Ox, the value ν_y was taken 0.8708 (see the table above). Here are the results compared to those communicated by de Smedt

x/a_1=	0.0833	0.2500	0.3333	0.5000	0.6667	0.7500	0.9167
de Smedt σ=	0.1143	0.3501	0.4759	0.7523	1.1460	1.4304	2.8182
Formula (33) σ=	0.1159	0.3577	0.4898	0.7999	1.2392	1.5709	3.1777
Discrepancy (%)	-1.3	-2.2	-2.9	-6.3	-8.1	-9.8	-12.8

We can also compare the same values along the axis Oy. One can use a formula similar to (33) replacing all x by y and interchanging a_1 and a_2, the value of ν_x was taken to be 0.3265.

y/a_2=	0.1667	0.3333	0.5000	0.6667	0.8333
de Smedt σ=	0.1756	0.3663	0.6011	0.9014	1.6413
our result σ=	0.1756	0.3673	0.5998	0.9292	1.5662
Discrepancy (%)	0.0	-0.3	0.2	-3.1	4.6

The agreement is satisfactory.

Rhombus. Let α be the angle at one of the rhombus apexes, and l be its side. Formulae (23–26) in this case yield

$$I_x = \frac{1}{6} l^4 \sin\alpha \sin^2\frac{\alpha}{2}, \qquad I_y = \frac{1}{6} l^4 \sin\alpha \cos^2\frac{\alpha}{2}, \qquad A = l^2 \sin\alpha,$$

$$J_x = 2l\sin\alpha\left[\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\ln\frac{\cos(\alpha/2)+\sin(\alpha/2)+1}{\cos(\alpha/2)+\sin(\alpha/2)-1}\right],$$

$$J_y = 2l\sin\alpha\left[-\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\ln\frac{\cos(\alpha/2)+\sin(\alpha/2)+1}{\cos(\alpha/2)+\sin(\alpha/2)-1}\right].$$

The coefficients will be defined as

$$\nu_x = \frac{8\sin^2\frac{\alpha}{2}}{9(\sin\alpha)^{3/2}\left[\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\ln\frac{\cos(\alpha/2)+\sin(\alpha/2)+1}{\cos(\alpha/2)+\sin(\alpha/2)-1}\right]},$$

$$\nu_y = \frac{8\cos^2\frac{\alpha}{2}}{9(\sin\alpha)^{3/2}\left[-\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\ln\frac{\cos(\alpha/2)+\sin(\alpha/2)+1}{\cos(\alpha/2)+\sin(\alpha/2)-1}\right]}. \qquad (3.4.34)$$

The same formulae in terms of the rhombus semiaxes a and b and the aspect ratio $\varepsilon = b/a$ has the form

$$\nu_x = \frac{2\sqrt{2\varepsilon}\,(1+\varepsilon^2)}{9\left[1-\varepsilon+\frac{\varepsilon^2}{\sqrt{1+\varepsilon^2}}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right]},$$

$$\nu_y = \frac{2\sqrt{2}\,(1+\varepsilon^2)}{9\varepsilon^{3/2}\left[\varepsilon-1+\frac{1}{\sqrt{1+\varepsilon^2}}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right]}. \qquad (3.4.35)$$

The coefficients of magnetic polarizability of a diamond were computed by de

Smedt (1979). Here, we present his results compared to those given by formula (35)

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x=	0.1181	0.1729	0.2341	0.3052	0.4101	0.4323	0.5193
Formula (35) ν_x=	0.1078	0.1627	0.2258	0.2986	0.4026	0.4230	0.5043
de Smedt ν_y=	6.1820	2.7060	1.5240	0.9946	0.6703	0.6323	0.5193
Formula (35) ν_y=	4.5987	2.1982	1.3254	0.9095	0.6388	0.6052	0.5043
Discrepancy of ν_x (%)	8.7	5.9	3.6	2.2	1.8	2.1	2.9
Discrepancy of ν_y (%)	25.6	18.8	13.0	8.6	4.7	4.3	2.9

The deterioration of the accuracy of (35) for small values of ε is the result of erroneous assumption of a square root singularity in (6) which is grossly incorrect for domains with sharp angles.

Circular segment. Let the radius r and the angle 2α be the segment parameters. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}. \tag{3.4.36}$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r[-k\cos\phi + (1 - k^2\sin^2\phi)^{1/2}] \quad \text{for} \quad 0 \le \phi \le \pi - \gamma \ \text{ or } \ \pi + \gamma \le \phi < 2\pi,$$

and

$$a(\phi) = r\,\frac{k - \cos\alpha}{\cos(\pi - \phi)} \qquad \text{for} \quad \pi - \gamma \le \phi \le \pi + \gamma. \tag{3.4.37}$$

Computation of the moments yields

$$A = r^2(\alpha - \tfrac{1}{2}\sin 2\alpha), \quad I_x = \tfrac{1}{4}Ar^2(1 - k\cos\alpha), \quad I_y = \tfrac{1}{4}Ar^2(1 + 3k\cos\alpha - 4k^2),$$

$$J_x = \frac{2}{3}r\left\{-k\sin^3\gamma + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\pi-\gamma, k)\right.$$

$$\left. + \frac{2k^2-1}{k^2}E(\pi-\gamma, k) + 3(k - \cos\alpha)\left[-\sin\gamma + \ln\tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right)\right]\right\},$$

$$J_y = \frac{2}{3} r \left\{ \sin\gamma \left[k\sin^2\gamma - 3\cos\alpha - (1 - k^2\sin^2\gamma)^{1/2}\cos\gamma \right] \right.$$

$$\left. - \frac{1-k^2}{k^2} F(\pi-\gamma, k) + \frac{1+k^2}{k^2} E(\pi-\gamma, k) \right\},$$

where $\gamma = \tan^{-1}(\sin\alpha/(k - \cos\alpha))$. Substituting in (22) leads to

$$\nu_x = \frac{4(1 - k\cos\alpha)}{\left[\alpha - \frac{1}{2}\sin 2\alpha\right]^{1/2}} \left\{ -k\sin^3\gamma + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2} F(\pi-\gamma, k) \right.$$

$$\left. + \frac{2k^2 - 1}{k^2} E(\pi-\gamma, k) + 3(k - \cos\alpha)\left[-\sin\gamma + \ln\tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right)\right] \right\}^{-1},$$

$$\nu_y = \frac{4(1 + 3k\cos\alpha - 4k^2)}{\left[\alpha - \frac{1}{2}\sin 2\alpha\right]^{1/2}} \left\{ \sin\gamma \left[k\sin^2\gamma - 3\cos\alpha - (1 - k^2\sin^2\gamma)^{1/2}\cos\gamma \right] \right.$$

$$\left. - \frac{1-k^2}{k^2} F(\pi-\gamma, k) + \frac{1+k^2}{k^2} E(\pi-\gamma, k) \right\}^{-1}. \qquad (3.4.38)$$

A plot of ν_x (full curve) and ν_y (broken curve) against the ratio α/π is given in Fig. 3.4.1. We are unaware of any data to verify the accuracy of (38).

Circular sector. Repetition of the procedure, described in the previous paragraph, leads to the following results for a circular sector with the angle 2α:

$$A = r^2\alpha, \quad I_x = \frac{1}{4} r^4 (\alpha - \frac{1}{2}\sin 2\alpha), \quad I_y = r^4 \frac{9\alpha^2 + 9\alpha\sin\alpha\cos\alpha - 16\sin^2\alpha}{36\alpha},$$

$$J_x = \frac{2}{3} r \left\{ -k\sin^3\gamma - (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2} F(\gamma, k) \right.$$

Fig. 3.4.1. Coefficients of magnetic polarizability for circular segment

$$+\frac{2k^2-1}{k^2}E(\gamma,k)+3k\sin\alpha\Big[\cos\alpha+\cos(\alpha+\gamma)$$

$$+\sin^2\alpha\ln\Big(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\Big)\Big]\Big\},$$

$$J_y=\frac{2}{3}r\Big\{k\sin\gamma(\sin^2\gamma-3)+(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma-\frac{1-k^2}{k^2}F(\gamma,k)$$

$$+\frac{1+k^2}{k^2}E(\gamma,k)+3k\sin\alpha\Big[-\cos\alpha-\cos(\alpha+\gamma)$$

$$+\cos^2\alpha\ln(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2})\Big]\Big\}. \tag{3.4.39}$$

Here, $k = 2\sin\alpha/(3\alpha)$, and $\gamma = \tan^{-1}(\sin\alpha/(\cos\alpha - k))$. The coefficients sought are expressed as follows

$$\nu_x = 2\alpha^{-3/2}(2\alpha - \sin 2\alpha)\Bigg\{ -k\sin^3\gamma - (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma$$

$$+ \frac{1-k^2}{k^2}F(\gamma, k) + \frac{2k^2-1}{k^2}E(\gamma, k) + 3k\sin\alpha\Bigg[\cos\alpha$$

$$+ \cos(\alpha+\gamma) + \sin^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\Bigg]\Bigg\}^{-1},$$

$$\nu_y = \frac{4(9\alpha^2 + 9\alpha\sin\alpha\cos\alpha - 16\sin^2\alpha)}{9\alpha^{5/2}}\Bigg\{ k\sin\gamma(\sin^2\gamma - 3)$$

$$+ (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma - \frac{1-k^2}{k^2}F(\gamma, k) + \frac{1+k^2}{k^2}E(\gamma, k)$$

$$+ 3k\sin\alpha\Bigg[-\cos\alpha - \cos(\alpha+\gamma) + \cos^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\Bigg]\Bigg\}^{-1}. \qquad (3.4.40)$$

Formulae (40) are exact for a complete circle ($\alpha = \pi$), and give the same results as (38) for a half-circle ($\alpha = \pi/2$). A plot of ν_x (full curve) and ν_y (broken curve) against the ratio α/π is given in Fig. 3.4.2.

Cross. Consider an aperture obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon = b/a$. The area and the moments will take the form

$$A = 4a^2\varepsilon(2-\varepsilon), \qquad I_x = I_y = \frac{4}{3}a^4\varepsilon(1 + \varepsilon^2 - \varepsilon^3),$$

$$J_x = J_y = 4a\left[\ln(\varepsilon + \sqrt{1+\varepsilon^2}) + \varepsilon\ln\frac{1+\sqrt{1+\varepsilon^2}}{(1+\sqrt{2})\varepsilon}\right].$$

The coefficients will be defined as

$$\nu_x = \nu_y = \frac{4\varepsilon(1+\varepsilon^2-\varepsilon^3)}{9\varepsilon(2-\varepsilon)^{3/2}}\left[\ln(\varepsilon + \sqrt{1+\varepsilon^2}) + \varepsilon\ln\frac{1+\sqrt{1+\varepsilon^2}}{(1+\sqrt{2})\varepsilon}\right]^{-1}. \qquad (3.4.41)$$

Fig. 3.4.2. Coefficients of magnetic polarizability for circular sector

The comparison between the results due to (41) and those given by de Smedt (1979) are presented below

$\varepsilon=$	0.1000	0.2000	0.3333	0.4000	0.5000	0.6000	0.7500	0.8000	1.0000
de Smedt $\nu_x=\nu_y=$	1.5910	0.8720	0.6255	0.5725	0.5267	0.5069	0.4985	0.4997	0.5193
Formula (41) $\nu_x=\nu_y=$	1.7382	0.8758	0.6006	0.5465	0.5049	0.4890	0.4893	0.4926	0.5043
Discrepancy (%)	-9.3	-0.4	4.0	4.5	4.1	3.5	1.9	1.4	2.9

Taking into consideration the shape complexity, we should consider the results agreement as surprisingly good, not only quantitatively but qualitatively as well: both data display a relatively flat minimum around $\varepsilon = 0.75$.

Variational approach. The accuracy of formulae (22) can be improved by taking into consideration the fifth harmonic (12) in combination with the variational approach (Noble 1960). The following functional assumes its maximum value at the exact solution of (1)

$$I(\sigma) = 2\iint_S \sigma(M)w(M)\,\mathrm{d}S_M - \iint_S \sigma(M)\left[\iint_S \frac{\sigma(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \qquad (3.4.42)$$

Taking

$$\iint_S \frac{\sigma(N)}{R(M,N)}\,\mathrm{d}S_N \approx w_1 + w_5, \qquad (3.4.43)$$

and substituting (6), (10), (12) and (43) in (42), one gets after integration with respect to ρ

$$I = \int_0^{2\pi} (a(\phi))^4 \Bigg\{ (p_1\cos\phi + p_2\sin\phi)\Bigg[\frac{4}{3}(\alpha_x\sin\phi - \alpha_y\cos\phi) - \frac{\pi}{3}(p_1J_y + p_2J_{xy})\cos\phi$$

$$-\frac{\pi}{3}(p_1J_{xy} + p_2J_x)\sin\phi - \frac{2\pi}{63}(a(\phi))^3([p_1(A_{c6} + A_{c4})$$

$$+ p_2(A_{s6} - A_{s4})]\cos 5\phi + [p_1(A_{s6} + A_{s4}) + p_2(A_{c4} - A_{c6})]\sin 5\phi)\Bigg]\Bigg\} \,d\phi. \tag{3.4.44}$$

Considering now the functional I as a function of p_1 and p_2, the extremum conditions

$$\frac{\partial I}{\partial p_1} = 0\,, \qquad \frac{\partial I}{\partial p_2} = 0$$

give two linear algebraic equations with respect to the unknowns p_1 and p_2. The complete solution is pretty cumbersome. Here, we present the final result for the coefficients ν_x and ν_y which are valid only for domains having at least one axis of symmetry, and the central principal axes taken as the coordinate axes

$$\nu_x = \frac{32I_x}{3A^{3/2}J_x(1+\eta_x)}\,, \qquad \nu_y = \frac{32I_y}{3A^{3/2}J_y(1+\eta_y)} \tag{3.4.45}$$

where the correction terms

$$\eta_x = \frac{(B_{c4} - B_{c6})(A_{c4} - A_{c6})}{84I_xJ_x}\,, \qquad \eta_y = \frac{(B_{c4} + B_{c6})(A_{c4} + A_{c6})}{84I_yJ_y}\,, \tag{3.4.46}$$

and

$$B_{c6} = \int_0^{2\pi} (a(\phi))^7 \cos 6\phi \,d\phi\,, \qquad B_{c4} = \int_0^{2\pi} (a(\phi))^7 \cos 4\phi \,d\phi\,.$$

Since expression (43) is approximate, there is no guarantee that (45) will be more accurate than (22). We performed the necessary computations for a rectangle. Here are the results compared to those by de Smedt (1979)

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt $\nu_x=$	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (45) $\nu_x=$	0.1403	0.1980	0.2558	0.3175	0.4165	0.4396	0.5510

de Smedt ν_y=	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (45) ν_y=	4.5294	2.0709	1.2355	0.8717	0.6606	0.6350	0.5510
Discrepancy in ν_x (%)	-9.0	-5.3	-1.1	2.3	1.8	0.9	-6.1
Discrepancy in ν_y (%)	-10.3	-2.2	1.9	2.0	-2.8	-3.6	-6.1

Comparison with similar data computed on the basis of formula (31) shows that the correction terms η_x and η_y in this particular case resulted in decreasing of the maximum value of discrepancy. We caution again that there is no guarantee that this will be valid for an arbitrary domain. The following rule of thumb may be suggested for the user wishing to improve the accuracy: when the value of the correction coefficients η_x and η_y does not exceed small percentage of unity this generally means an improvement in accuracy, otherwise one should not use formulae (45).

It is worthwhile to give the solution due to (44) for the case when the aperture has no axis of symmetry, and only the first harmonic of w_1 is taken into consideration. The result is

$$p_1 = \frac{\alpha_x(c_{22}I_{xy} - c_{12}I_x) + \alpha_y(c_{12}I_{xy} - c_{22}I_y)}{c_{11}c_{22} - c_{12}^2},$$

$$p_2 = \frac{\alpha_x(c_{11}I_x - c_{12}I_{xy}) + \alpha_y(c_{12}I_y - c_{11}I_{xy})}{c_{11}c_{22} - c_{12}^2}, \tag{3.4.47}$$

where

$$c_{11} = \frac{\pi}{2}(J_y I_y + J_{xy}I_{xy}), \quad c_{22} = \frac{\pi}{2}(J_x I_x + J_{xy}I_{xy}),$$

$$c_{12} = \frac{\pi}{4}(J_{xy}(I_x + I_y) + I_{xy}(J_x + J_y)).$$

Formulae (47) look different from the equivalent set (14) derived earlier. In the absence of any numerical data related to a general domain, it is impossible to say whether formulae (47) are more accurate than (14), but they are definitely more complicated. It is noteworthy that in the case of a domain with an axis of symmetry both (47) and (14) simplify to the same equations (17).

3.5. Electrical polarizability of small apertures

The method of previous section is used here for analytical solution to the problem of electrical polarizability of arbitrarily shaped small apertures. A simple general formula is established for the computation of the coefficients of electrical polarizability. Specific formulae are derived for apertures of various shapes. A comparison is made with the results available in the literature.

At the present time closed form expression for electric polarizability is known for an elliptic aperture in a planar screen only. Some specific shapes were treated either experimentally (Cohn 1952) or numerically (Okon and Harrington 1981, De Meulenaere and Van Bladel 1977). A variational approach to the problem was proposed by Fikhmanas and Fridberg (1973) who also suggested an empirical formula for evaluating the coefficients of electrical polarizability which will be discussed further. No analytical approach for the case of non-elliptical apertures has been reported as yet. The first attempt to do so was made in (Fabrikant, 1987d).

Consider a flat screen conceived as the plane z=0, with an electrically small aperture S whose boundary is given in the polar coordinates as

$$\rho = a(\phi). \tag{3.5.1}$$

It is well known (*e.g.* De Meulenaere and Van Bladel 1977) that the governing integral equation for the electric polarizability density w can be written in the form

$$\sigma(N) = \Delta \int\!\!\int_S \frac{w(M)}{R(M,N)}\,\mathrm{d}S. \tag{3.5.2}$$

where Δ is the two-dimensional Laplace operator, S is the aperture domain, $R(M,N)$ stands for the distance between the points M and N, σ is a known function ($\sigma=-2\pi/\sqrt{A}$, where A is the aperture area). We use again the integral representation

$$\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\,\mathrm{d}x}{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}. \tag{3.5.3}$$

Substitution of (3) into (2) gives, after changing the order of integration

$$\sigma(\rho,\phi) = \frac{2}{\pi}\Delta\int_0^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_0^{2\pi}\mathrm{d}\phi_0\int_x^{a(\phi_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)}{\sqrt{\rho_0^2-x^2}}\, w(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0. \tag{3.5.4}$$

Again we have to state that the change of the order of integration which led to (4) is valid inside the circle $\rho\le min\{a(\phi)\}$ only. Nevertheless, its usefulness will be demonstrated quite convincingly.

Let the electrical polarizability density w be

$$w = \frac{\delta}{a(\phi)} \sqrt{a^2(\phi) - \rho^2}, \tag{3.5.5}$$

where δ is a constant to be defined. Now substituting (5) in (4), we can verify how close to a constant will be the value of σ produced by (5). Integration with respect to ρ_0 gives

$$\sigma(\rho, \phi) = \frac{\delta}{2} \sum_{n=-\infty}^{\infty} \Delta \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x\,dx}{\sqrt{\rho^2 - x^2}} \int_0^{2\pi} \frac{a^2(\phi_0) - x^2}{a^2(\phi_0)}$$

$$\times F\left(2 - \frac{|n|}{2}, \frac{1}{2}; 2; 1 - \frac{x^2}{a^2(\phi_0)}\right) e^{in(\phi - \phi_0)}\, d\phi_0. \tag{3.5.6}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the value of σ can be done separately for each harmonic. The zero*th* harmonic has the form

$$\sigma_0 = \frac{\pi\delta G}{2}, \tag{3.5.7}$$

where the notation was introduced

$$G = \int_0^{2\pi} \frac{d\phi}{a(\phi)}. \tag{3.5.8}$$

It is quite clear that the integral value in (8) will depend not only on the domain contour but also on the location of the system of coordinate origin. The following logic might be useful for establishing certain rule in this regard. According to (5), the coordinate origin location corresponds to the point where the electrical polarizability density attains its maximum. We shall call this point *the aperture center*. In the case of an aperture domain with one axis of symmetry, we may conclude from physical considerations that this point should be located at the axis. When this domain possesses two axes of symmetry the location of the aperture center is at their intersection, e.g. at the center of gravity of the domain. It is noteworthy that the integral (8) attains its minimum in this case. One can extend this rule to a general aperture, namely, the aperture center should be identified with the point inside S where the integral (8) reaches its minimum. Direct computations for various domains indicate that this minimum is, in general, sufficiently flat, so that in many cases one may locate the aperture center at the center of gravity, without significant loss in accuracy.

It is important to note that the second harmonic is equal to zero for an arbitrary contour, and that all the odd harmonics will be zero if the expression for $a(\phi)$ does not contain odd harmonics. Here is the expression for the fourth harmonic

$$\sigma_4 = \frac{16}{5}\delta\rho \int_0^{2\pi} \frac{\cos 4(\phi - \phi_0)\, d\phi_0}{a^2(\phi_0)}. \tag{3.5.9}$$

The investigation of further harmonics shows that their amplitude decreases for general domains, and they vanish in the case of an ellipse. If we assume that $\sigma_0 \approx -2\pi/\sqrt{A}$ then the remaining harmonics may be called the solution error. This means establishment of the following relationship

$$-\frac{2\pi}{\sqrt{A}} = -\frac{\pi\delta G}{2}. \tag{3.5.10}$$

where A is the aperture area. Immediate consequence of (10) is

$$\delta = \frac{4}{G\sqrt{A}}. \tag{3.5.11}$$

One can verify that in the case of an ellipse, the solution given by (5) and (11) is *exact*. We expect it to be reasonably accurate for an aperture of general shape. This assumption will be justified later on. We also expect (5) to be sufficiently accurate in the neighborhood of the aperture center, though the relative error might be quite significant close to the boundary.

Introduce the coefficient of electrical polarizability τ as the average

$$\tau = \frac{1}{A}\int\int_S w\, dS. \tag{3.5.12}$$

Substitution of (5) in the last expression yields

$$\tau = \frac{2}{3}\delta, \tag{3.5.13}$$

which gives, after comparison with (11)

$$\tau = \frac{8}{3\sqrt{A}G}. \tag{3.5.14}$$

One can deduce that the value of τ does not depend on the size of the domain S, and is defined by its shape only. It attains its maximum in the case of a circle, so that $0 \le \tau \le 4/(3\pi^{3/2}) = 0.2394$. Tabulation of the coefficient τ for various aperture shapes might prove very useful since its knowledge allows to find the maximum (or average) value of the electric polarizability density. An empirical formula for the coefficient of electrical polarizability was proposed by Fikhmanas and Fridberg (1973). This formula in our notation reads

$$\tau = \frac{8\sqrt{A}}{3\pi L} \tag{3.5.15}$$

where L stands for the perimeter of the domain S. Formula (15) is also exact for an ellipse. It would be interesting to compare its performance with our (14). Several aperture shapes are considered below.

Polygon. Consider a polygon with n sides, with the only limitation that the function $a(\phi)$ describing its boundary be continuous and single-valued. The origin of the coordinate system is located at the aperture center as it is defined above. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered k+1. It is clear that the value of index equal n+1 is understood as 1. Denote b_k the distance from the aperture center to the kth apex. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. Then formulae (8) and (14) yield the following expression for the coefficient τ

$$\tau = \frac{8}{3\sqrt{A}}\left\{\sum_{k=1}^{n}\left[\frac{a_k^2}{4A_k^2} - \frac{1}{b_k^2}\right]^{1/2} + \left[\frac{a_k^2}{4A_k^2} - \frac{1}{b_{k+1}^2}\right]^{1/2}\right\}^{-1}. \tag{3.5.16}$$

In the case of a regular polygon formula (16) simplifies to

$$\tau = \frac{4\sqrt{\cot(\pi/n)}}{3n^{3/2}\sin(\pi/n)}. \tag{3.5.17}$$

Formula (15) gives for a regular polygon

$$\tau = \frac{4}{3\pi}\left[\frac{\cot(\pi/n)}{n}\right]^{1/2} \tag{3.5.18}$$

It is of interest to compare the numerical results due to (17) and (18). Here the relevant computations are presented

$n=$	3	4	5	6	9	100
formula (17) $\tau=$	0.2251	0.2357	0.2380	0.2388	0.2393	0.2394
formula (18) $\tau=$	0.1862	0.2122	0.2227	0.2280	0.2345	0.2394
discrepancy %	17.3	10.0	6.5	4.5	2.0	0.0

While both formulae in the limiting case $n\to\infty$ give the same exact result for a circle, their discrepancy for small n is quite significant, so it is important to establish which one is more accurate. We did not find any reliable data for equilateral triangle. If one takes the experimental result by Cohn for a square $\tau=0.2274$ as exact, then our formula (17) is in error by 3.6% while formula (18) due to Fikhmanas and Fridberg is in error by 6.7%. The numerical result due to Okon and Harrington for a square is 0.2258 which also favors our formula. In the case of a regular hexagon, the result by Okon and Harrington is 0.2375, so that our result is about 0.5% away while the error of (18) is 4%. It is noteworthy that the value of τ does not change significantly in the whole range $3\leq n<\infty$.

We can also compare the electric polarizability density distribution along a central line of a hexagon perpendicular to its side, given by Okon and Harrington (1981), with those due to (5). Here are the data

$\rho/a=$	0.	0.1667	0.3333	0.5000	0.6667	0.8333
Okon *et al* $w=$	0.351	0.346	0.331	0.305	0.263	0.210
formula (5) $w=$	0.357	0.352	.3366	0.3092	0.2660	0.1973
discrepancy %	-1.7	-1.7	-1.4	-1.4	-1.2	6.0

As we expected, the agreement is good, except for the points very close to the boundary.

Rectangle. Consider a rectangular aperture, a and b being its semiaxes along the axes Ox and Oy respectively. Introduce the aspect ratio $\varepsilon=b/a\leq 1$. Formula (17) in this case reduces to

$$\tau=\frac{\sqrt{\varepsilon}}{3(1+\varepsilon^2)^{1/2}}. \tag{3.5.19}$$

Formula (15) in this case gives

$$\tau=\frac{4\sqrt{\varepsilon}}{3\pi(1+\varepsilon)} \tag{3.5.20}$$

We present below the results of computations due to (19) and (20) compared with the experimental results by Cohn (1952)

$\varepsilon=$	0.1000	0.1500	0.2000	0.3000	0.5000	0.7500	1.0000
experiment $\tau=$	0.1202	0.1411	0.1565	0.1789	0.2093	0.2251	0.2274
formula (19) $\tau=$	0.1049	0.1277	0.1462	0.1749	0.2108	0.2309	0.2357
discrepancy %	12.7	9.5	6.6	2.3	-0.7	-2.6	-3.7

formula (20) τ=	0.1220	0.1429	0.1582	0.1788	0.2001	0.2100	0.2122
discrepancy %	-1.5	-1.3	-1.1	0.1	4.4	6.7	6.7

If one assumes the results by Cohn as exact then our formula performs better for $\varepsilon \geq 0.5$ while the formula by Fikhmanas and Fridberg is more accurate for $\varepsilon < 0.5$. If we take the numerical results received in a personal communication from De Smedt as correct then the conclusion might be different. For example, his value of τ for $\varepsilon=0.1$ is 0.1142; now our result is in error by 8% while the result by Fikhmanas and Fridberg is in error by -7%. At this moment nobody seems to know which estimation is correct. We can also compare the distribution of w due to (5) with the numerical results received in a personal communication from De Smedt for a rectangle with the aspect ratio $\varepsilon=0.5$. Here are the data computed along the axis Ox for $y/b=0.025$.

x/a=	0.0250	0.2250	0.4250	0.6250	0.8250	0.9750
De Smedt w=	0.3161	0.3118	0.2989	0.2713	0.2107	0.0852
formula (5) w=	0.3158	0.3081	0.2862	0.2469	0.1787	0.0703
Discrepancy %	0.1	1.2	4.2	9.0	15.2	17.5

The agreement is not bad except for the zone close to the boundary. Here are the data computed along the axis Oy for $x/a=0.025$.

y/b=	0.0250	0.1250	0.2250	0.3250	0.4250	0.4750
De Smedt w=	0.3161	0.3067	0.2836	0.2424	0.1690	0.0976
formula (5) w=	0.3158	0.3062	0.2824	0.2403	0.1666	0.0987
Discrepancy %	0.1	0.2	0.4	0.8	1.4	-1.2

We observe here a good agreement even close to the boundary.

Rhombus. Consider the case when the domain S is a rhombus, a and b being its semiaxes along the axes Ox and Oy respectively. Introduce the aspect ratio $\varepsilon=b/a\leq 1$. Formula (16) in this case reduces to

$$\tau = \frac{\sqrt{2\varepsilon}}{3(1+\varepsilon)}. \tag{3.5.21}$$

The result due to Fikhmanas and Fridberg is

$$\tau = \frac{2\sqrt{2\varepsilon}}{3\pi(1+\varepsilon^2)^{1/2}}. \tag{3.5.22}$$

The coefficient of electrical polarizability for a diamond with the aspect ratio $\varepsilon=0.5$ was found numerically by Okon and Harrington as $\tau=0.2082$. Our result is 0.2222 (discrepancy 6.7%) while formula (22) gives 0.1898 (discrepancy 8.9%). We have received two sets of data in personal communications from De Smedt and Lee. Here are the data received as compared to formulae (21) and (22)

ε=	0.100	0.200	0.333	0.500	0.800	1.000

De Smedt τ=	0.111	0.151	0.182	0.204	0.219	0.221
formula (21) τ=	0.136	0.176	0.204	0.222	0.234	0.236
Discrepancy %	-21.9	-16.4	-12.0	-9.0	-6.8	-6.6
formula (22) τ=	0.094	0.132	0.164	0.190	0.210	0.212
Discrepancy %	15.1	12.8	9.8	6.9	4.4	4.1

The data received from Lee is given as a function of the angle $\alpha = \tan^{-1}\varepsilon$

α(deg.)=	10.0	15.0	20.0	25.0	30.0	40.0	45.0
Lee τ=	0.147	0.174	0.193	0.207	0.216	0.226	0.228
formula (21) τ=	0.168	0.192	0.209	0.220	0.227	0.235	0.236
Discrepancy %	-14.2	-10.6	-8.1	-6.3	-5.2	-3.8	-3.6
formula (22) τ=	0.124	0.150	0.170	0.186	0.197	0.211	0.212
Discrepancy %	15.8	13.7	11.8	10.1	8.5	6.9	6.7

We have presented both sets of data in order to underline the fact that there is no really reliable data as yet. The first set of data makes the formula by Fikhmanas and Fridberg more accurate, while the second set favors ours. It is noteworthy that formula (21) seems to give the upper bound, and formula (22) provides the lower bound, their average being very close to the numerical data.

We can also compare the value of electrical polarizability density due to our (5) with a similar result due to Okon and Harrington (1981). Here are the data computed along a central line parallel to its side

ρ/a=	.0	0.3333	0.6667
Okon *et al.* w=	0.335	0.304	0.257
formula (5) w=	0.3333	0.3142	0.2484
discrepancy %	0.5	-3.4	3.3

The agreement is worse if the comparison is made along the major axis. This is mainly due to the assumption of a square root singularity in (5) which does not hold for domains with sharp angles.

Circular segment. Let the radius r and the angle 2α be the segment parameters. Direct numerical computations show that the aperture center can be identified with the center of gravity, with the error comparable with the accuracy of the theory presented. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}.$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r[-k\cos\phi + (1-k^2\sin^2\phi)^{1/2}] \quad \text{for} \quad 0 \le \phi \le \pi - \gamma \ \text{ or } \ \pi + \gamma \le \phi < 2\pi,$$

and

$$a(\phi) = r\,\frac{k-\cos\alpha}{\cos(\pi-\phi)} \qquad \text{for} \quad \pi-\gamma \le \phi \le \pi+\gamma. \tag{3.5.23}$$

Feeding of (23) in (8) and (14) gives

$$\tau = \frac{4}{3(\alpha-\frac{1}{2}\sin 2\alpha)^{1/2}}\left[\frac{k\sin\gamma + E(\pi-\gamma, k)}{1-k^2} + \frac{\sin\gamma}{k-\cos\alpha}\right]^{-1}, \tag{3.5.24}$$

where $\gamma = \tan^{-1}(\sin\alpha/(k - \cos\alpha))$. The formula due to Fikhmanas and Fridberg will take the form

$$\tau = \frac{4(\alpha-\frac{1}{2}\sin 2\alpha)^{1/2}}{3\pi(\alpha+\sin\alpha)}. \tag{3.5.25}$$

The coefficient of electrical polarizability for a semi-circle was computed by Okon and Harrington as $\tau=0.2161$. Our result due to (24) is $\tau=0.2163$ which is practically identical to the previously mentioned one. The result due to (25) is $\tau=0.2069$ (discrepancy 4.3%). An additional confirmation of correctness of the new method can be obtained by observing the plot of the electrical polarizability density distribution for a semi-circle presented by Okon and Harrington (1981). Its maximum is located at the distance $\approx 0.47r$ from the circle's center. Our definition of the aperture center requiring the minimization of the integral (8) gives its coordinate at $0.48r$ which is very close. The center of gravity of the semi-circle is located at $0.42r$. Figure 3.5.1 plots the value of τ against α/π due to formulae (24) (solid line) and (25) (broken line).

Circular sector. Let r and 2α be its radius and the polar angle. The aperture center is assumed to be located on the axis of symmetry at a distance kr from the circle's center. Direct computations show that the aperture center may be located at the center of gravity for $0.1\pi<\alpha<0.6\pi$. In this case the value of k is defined by $k=2\sin\alpha/(3\alpha)$. In the range $\alpha<0.1\pi$ or $\alpha>0.6\pi$, the value of k should be found from the minimum condition for the integral (8). Repetition of the procedure, described in the previous paragraph, leads to the following result

$$\tau = \frac{4}{3\sqrt{\alpha}}\left[\frac{k\sin\gamma + E(\gamma, k)}{1-k^2} + \frac{\cos\alpha + \cos(\alpha-\gamma)}{k\sin\gamma}\right]^{-1}. \tag{3.5.26}$$

Here, $\gamma = \tan^{-1}(\sin\alpha/(\cos\alpha - k))$. The formula due to Fikhmanas and Fridberg reads

Fig. 3.5.1. Coefficient of electrical polarizability for circular segment

$$\tau = \frac{4\sqrt{\alpha}}{3\pi(1+\alpha)}. \tag{3.5.27}$$

Note that neither (26) nor (27) reduce to the exact value for a circle when $\alpha=\pi$. This is due to the fact that we do not really have the case of a complete circular aperture when α approaches π: we have a circular aperture which has its radius $\phi=\pi$ grounded. This case has not been considered by other authors so we can not say which formula is more accurate. Okon and Harrington in the case of a quadrant obtained $\tau=0.2269$, formula (26) gives $\tau=0.2308$ (discrepancy 1.7%), and formula (27) gives $\tau=0.2107$ (discrepancy 7%). Figure 3.5.2 plots the value of τ against α/π due to formulae (26) (solid line) and (27) (broken line).

Cross. Consider an aperture configuration obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon = b/a \le 1$. The area can be expressed as

$$A = 4a^2\varepsilon(2-\varepsilon),$$

The following expression can be obtained for τ, namely,

Fig. 3.5.2. Coefficient of electrical polarizability for circular sector

$$\tau = \frac{\sqrt{2\varepsilon}}{6\sqrt{2-\varepsilon}\,\{[2(1+\varepsilon^2)]^{1/2}-1\}}\,. \tag{3.5.28}$$

The formula due to Fikhmanas and Fridberg is

$$\tau = \frac{2\sqrt{\varepsilon(2-\varepsilon)}}{3\pi}\,. \tag{3.5.29}$$

Here, we present the results given by formulae (28) and (29) compared to the experimental results by Cohn (1952) and the numerical results by De Meulenaere and Van Bladel (1977), and those received in personal communication from De Smedt

ε=	0.1000	0.2000	0.3000	0.4000	0.6000	0.8000	1.0000
experimental τ=	0.0942	0.1333	0.1609	–	–	–	0.2274
De Meulenaere τ=	–	–	–	0.19	0.22	0.23	0.238
De Smedt τ=	0.0835	0.1183	–	0.1767	0.2084	0.2193	0.2212
formula (28) τ=	0.1284	0.1777	0.2078	0.2252	0.2376	0.2372	0.2357
formula (29) τ=	0.0925	0.1273	0.1515	0.1698	0.1944	0.2079	0.2122

We did not compute the discrepancy since the data disagreement is too big thus making all the data not very reliable. The general impression is that our (28) gives the upper bound for τ while the formula due to Fikhmanas and Fridberg provides the lower bound. This conclusion might be wrong if the numerical results received in the personal communication from De Smedt are correct. For example, his result for $\varepsilon=0.1$ is $\tau=0.08347$ which differs from the experimental result by 11%. All this proves one point: the existing numerical methods are too crude, and there is a need to develop some new and more reliable numerical methods. It should be noted that the function defined by (28) is not monotonic: a relatively flat maximum is observed for $\varepsilon\approx0.7$. The remaining data are monotonic. We have no rigorous proof to claim that the quantitative behavior of (28) is correct while the other data behavior is not, but we can indicate that the value of τ for a quadrant is also greater than that for a semi-circle, and this is mainly due to the fact that the shape of a quadrant is more close to a circle than the shape of a semi-circle. Similar statement can be made about a cross with the aspect ratio $\varepsilon\approx0.7$ as compared to a square.

Majority of the examples considered indicate that the exact result is sandwiched between those given by our (14) and by the formula due to Fikhmanas and Fridberg (15). In this sense both formulae act as an upper bound and a lower bound which leads to a conjecture: for a certain class of contours one of the inequalities holds, namely, either $\tau_{14}\le\tau_{exact}\le\tau_{15}$, or $\tau_{14}\ge\tau_{exact}\ge\tau_{15}$. A significant effort is required to find such class of contours for which one of the conjectures holds, and it is beyond the scope of this book.

The accuracy of formula (14) can be improved in some cases by taking into consideration the fourth harmonic (9) in combination with the variational approach (Noble 1960). The following functional assumes its stationary value at the exact solution of (2)

$$I(w)=2\iint_S \sigma(M)w(M)\mathrm{d}S_M-\iint_S w(M)\left[\Delta\iint_S \frac{w(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \tag{3.5.30}$$

Taking

$$\Delta\iint_S \frac{w(N)}{R(M,N)}\mathrm{d}S_N\approx\sigma_0+\sigma_4$$

where σ_0 and σ_4 are defined by (7) and (9) respectively, and substituting them in (30), we obtain a functional which can be considered as a function of δ. From the extremum condition

$$\frac{\partial I}{\partial\delta}=0.$$

one finally gets

$$\tau = \frac{8}{3G\sqrt{A}(1-\eta)}, \tag{3.5.31}$$

where

$$\eta = \frac{3(F_c E_c + F_s E_s)}{5AG},$$

and the following geometrical characteristics were introduced

$$F_c = \int_0^{2\pi} \frac{\cos 4\phi\, d\phi}{a^2(\phi)}, \quad F_s = \int_0^{2\pi} \frac{\sin 4\phi\, d\phi}{a^2(\phi)},$$

$$E_c = \int_0^{2\pi} a^3(\phi) \cos 4\phi\, d\phi, \quad E_s = \int_0^{2\pi} a^3(\phi) \sin 4\phi\, d\phi,$$

The results of computations due to (31) for a rectangle are presented below against the experimental results by Cohn

ε=	0.1000	0.1500	0.2000	0.3000	0.5000	0.7500	1.0000
Cohn τ=	0.1202	0.1411	0.1565	0.1789	0.2093	0.2251	0.2274
formula (31) τ=	0.1054	0.1290	0.1484	0.1785	0.2125	0.2257	0.2278
discrepancy %	12.3	8.6	5.2	0.2	-1.5	-0.3	-0.2

Comparison of this table with a similar one above indicates that the variational approach does improve the accuracy, though the improvement is still not sufficient for small ε. This example can not be considered as a proof of better accuracy of the variational approach. We are quite sure that one can produce examples showing the opposite. It is up to the user to decide whether the more cumbersome computations are worth somewhat better accuracy.

3.6. Dirichlet problem for an annular disk

It is impossible even to mention all the publications related to the Dirichlet problem for a flat circular annulus. Their number is awesome. Tranter (1960) and Gubenko (1960) were among the first to consider the problem. One can find many references related to contact problem in (Borodachev, 1976), other references related to the equivalent electrostatic problem can be found in Love (1976). Majority of publications is devoted to the axisymmetric problems. Though some results related to consideration of specific harmonics have been published (Williams, 1963; Cooke, 1963), no general solution to the problem has been attempted as yet. This kind of solution is now possible. The problem is reduced to a set of two two-dimensional Fredholm integral equations with an elementary non-singular kernel which can be solved by iterations. This set can be easily uncoupled. The case of conducting circular annulus kept at constant potential and the problem of magnetic polarizability of such a disk are considered as examples. The governing integral equations are solved exactly in series involving the iterated kernels. Approximate formulae are derived for the case of a wide annulus.

Theory. It is convenient to reformulate the Dirichlet problem for a circular annulus as a mixed boundary value problem of potential theory for a half space $z \geq 0$. We need to find a harmonic function V vanishing at infinity and satisfying the following conditions at $z = 0$:

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for} \quad b < \rho < a, \quad 0 \leq \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for} \quad \rho < b \quad \text{or} \quad \rho > a, \quad 0 \leq \phi < 2\pi. \tag{3.6.1}$$

Here v is a known function. The approach proposed here is inspired by the elegant solution for the capacity of an annulus (Love, 1976) which is based on the method described in (Clement and Love, 1974) for solving axisymmetric problems. It looked very challenging to generalize the approach for non-axisymmetric case. Such a generalization has been found after several trials and errors, and it is presented here. The general approach is based on the results presented in Chapter 1. Let us introduce two harmonic functions

$$V_1(\rho,\phi,z) = -\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{b} \frac{\sqrt{\rho_0^2 - l_1^2(\rho_0)}}{R_0^2} f_1(\rho_0,\phi_0)\, d\rho_0 d\phi_0$$

$$= -\frac{2}{\pi}\int_0^{b} \frac{\sqrt{\rho_0^2 - l_1^2(\rho_0)}}{l_2^2(\rho_0) - l_1^2(\rho_0)}\, \mathcal{L}\left(\frac{l_1(\rho_0)}{l_2(\rho_0)}\right) f_1(\rho_0,\phi)\, d\rho_0; \tag{3.6.2}$$

$$V_2(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} \frac{\sqrt{l_2^2(\rho_0) - \rho_0^2}}{R_0^2} f_2(\rho_0,\phi_0)\, d\rho_0 d\phi_0$$

$$= \frac{2}{\pi}\int_a^{\infty} \frac{\sqrt{l_2^2(\rho_0) - \rho_0^2}}{l_2^2(\rho_0) - l_1^2(\rho_0)}\, \mathcal{L}\left(\frac{l_1(\rho_0)}{l_2(\rho_0)}\right) f_2(\rho_0,\phi)\, d\rho_0. \tag{3.6.3}$$

Here f_1 and f_2 are the as yet unknown functions, and the following notations were introduced:

$$l_1(x) = \frac{1}{2}\{\sqrt{(\rho + x)^2 + z^2} - \sqrt{(\rho - x)^2 + z^2}\},$$

$$l_2(x) = \frac{1}{2}\{\sqrt{(\rho + x)^2 + z^2} + \sqrt{(\rho - x)^2 + z^2}\},$$

$$R_0 = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2}. \tag{3.6.4}$$

We remind that the $\mathcal{L}$-operator for $k<1$ is understood as follows:

$$\mathcal{L}(k)f(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi-\phi_0)\, f(\rho,\phi_0)\, d\phi_0$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|} e^{in\phi} \int_0^{2\pi} e^{-in\phi_0} f(\rho,\phi_0)\, d\phi_0 = \sum_{n=-\infty}^{\infty} k^{|n|} f_n(\rho) e^{in\phi}. \tag{3.6.5}$$

Here f_n is the n-th Fourier coefficient of the function f, and

$$\lambda(k,\psi) = \frac{1 - k^2}{1 - 2k\cos\psi + k^2}. \tag{3.6.6}$$

One can easily verify that the potential in (2) vanishes on the plane $z=0$ for $\rho>b$, while the potential in (3) vanishes on the boundary for $\rho<a$. These properties allow us to reformulate the problem as the Dirichlet problem for a half space, with the potential prescribed all over the plane $z=0$, namely,

$$V(\rho,\phi,0) = V_1(\rho,\phi,0), \quad \text{for} \quad 0 \le \rho < b, \quad 0 \le \phi < 2\pi;$$

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for} \quad b \le \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$V(\rho,\phi,0) = V_2(\rho,\phi,0), \quad \text{for} \quad a < \rho < \infty, \quad 0 \le \phi < 2\pi. \tag{3.6.7}$$

Thus, the first boundary condition in (1) is satisfied. The unknown functions f_1 and f_2 are to be chosen in such a way that the second boundary condition in (1) is satisfied too.

Formulae (2) and (3) on the plane $z=0$ take the form

$$V_1(\rho,\phi,0) = -\frac{2}{\pi}\int_\rho^b \mathcal{L}\left(\frac{\rho}{\rho_0}\right)\frac{f_1(\rho_0,\phi)\,d\rho_0}{\sqrt{\rho_0^2-\rho^2}},$$

$$V_2(\rho,\phi,0) = \frac{2}{\pi}\int_a^\rho \mathcal{L}\left(\frac{\rho_0}{\rho}\right)\frac{f_2(\rho_0,\phi)\,d\rho_0}{\sqrt{\rho^2-\rho_0^2}}. \tag{3.6.8}$$

On the other hand, from (1.3.47) by assuming $z=0$ the following expression can be obtained for the potential which is nonzero in the interval $[0,b]$ and zero outside this interval:

$$V_1(\rho,\phi,0) = 4\int_\rho^b \frac{dx}{\sqrt{x^2-\rho^2}}\int_0^x \frac{\rho_0\,d\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_1(\rho_0,\phi). \tag{3.6.9}$$

Here σ_1 denotes the charge density distribution. Comparison of (7) and (9) yields the following relationship between σ_1 and f_1:

$$f_1(\rho,\phi) = -2\pi\int_0^\rho \frac{\rho_0\,d\rho_0}{\sqrt{\rho^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma_1(\rho_0,\phi). \tag{3.6.10}$$

The inverse relationship is readily available, and is

$$\sigma_1(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\min(\rho,b)} \frac{\rho_0\,d\rho_0}{\sqrt{\rho^2-\rho_0^2}}\mathcal{L}(\rho_0)f_1(\rho_0,\phi). \tag{3.6.11}$$

By comparing in the same manner the expression (1.4.33) for $z=0$

$$V_2(\rho,\phi,0) = 4\int_a^\rho \frac{dx}{\sqrt{\rho^2-x^2}}\int_x^\infty \frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma_2(\rho_0,\phi), \tag{3.6.12}$$

with (8), we obtain

$$f_2(\rho,\phi) = 2\pi\int_\rho^\infty \frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma_2(\rho_0,\phi). \tag{3.6.13}$$

The inverse to (13) takes the form

$$\sigma_2(\rho,\phi) = -\frac{\mathcal{L}(\rho)}{\pi^2\rho}\frac{d}{d\rho}\int_{\max(\rho,a)}^{\infty}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)f_2(\rho_0,\phi). \tag{3.6.14}$$

We have presented the resultant field as a superposition of three fields, namely, the field due to potential equal to $V_1(\rho,\phi,0)$ on the interval $0\le\rho\le b$, and zero outside the interval; the field due to potential equal to $v(\rho,\phi)$ in the interval $b\le\rho\le a$, and zero outside this interval; and the field due to potential equal to $V_2(\rho,\phi,0)$ on the interval $a\le\rho<\infty$, and zero outside the interval. The corresponding charge density distributions will be denoted as σ_1, σ_0, and σ_2 respectively. Now we can use the fact that the total charge $\sigma=\sigma_1+\sigma_0+\sigma_2=0$ in the intervals $\rho\le b$ and $\rho\ge a$. These conditions will give us two equations from which the as yet unknown functions f_1 and f_2 can be found. By using the result established in (Fabrikant, 1989a), we can write

$$\sigma_0(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{x\,dx}{\sqrt{\rho^2-x^2}}\mathcal{L}(x^2)\frac{d}{dx}\int_b^{a}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi), \quad \text{for} \quad \rho\le b. \tag{3.6.15}$$

By using (11), (14), and (15) the following equation may be written for $\rho\le b$:

$$-\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{\rho_0\,d\rho_0}{\sqrt{\rho^2-\rho_0^2}}\mathcal{L}(\rho_0)f_1(\rho_0,\phi) - \frac{\mathcal{L}(\rho)}{\pi^2\rho}\frac{d}{d\rho}\int_a^{\infty}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)f_2(\rho_0,\phi)$$

$$-\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{x\,dx}{\sqrt{\rho^2-x^2}}\mathcal{L}(x^2)\frac{d}{dx}\int_b^{a}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi) = 0. \tag{3.6.16}$$

Application of an operator

$$\int_0^{r}\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}\left(\frac{\rho}{r}\right)$$

to both sides of (16) yields

$$\frac{1}{2\pi} f_1(r,\phi) + \frac{1}{\pi^2}\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}} \int_a^\infty \frac{\rho_0\, d\rho_0}{(\rho_0^2-\rho^2)^{3/2}} \mathcal{L}\left(\frac{\rho^2}{\rho_0 r}\right) f_2(\rho_0,\phi)$$

$$+ \frac{1}{2\pi} r \int_b^a \frac{\rho_0\, d\rho_0}{(\rho_0^2-r^2)^{3/2}} \mathcal{L}\left(\frac{r}{\rho_0}\right) v(\rho_0,\phi) = 0. \tag{3.6.17}$$

We can interchange the order of integration in the second term of (17) and perform the integration with respect to ρ. Then equation (17) will take the form

$$f_1(\rho,\phi) + \frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty Q_1(\rho,\rho_0,\phi-\phi_0)\, f_2(\rho_0,\phi_0)\, d\rho_0\, d\phi_0 = g_1(\rho,\phi), \tag{3.6.18}$$

where

$$g_1(\rho,\phi) = -\rho\int_b^a \frac{\rho_0\, d\rho_0}{(\rho_0^2-\rho^2)^{3/2}} \mathcal{L}\left(\frac{\rho}{\rho_0}\right) v(\rho_0,\phi), \tag{3.6.19}$$

$$Q_1(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{\frac{\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}}}{(\rho_0 e^{i(\phi-\phi_0)}-\rho)R} \tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\rho/\rho_0)}{(\rho_0/\rho)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\} - \frac{\rho}{R^2}, \tag{3.6.20}$$

with

$$R = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{3.6.21}$$

Here the following integral was used

$$\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}\,(y^2-\rho^2)^{3/2}(1-m\rho^2)} = \frac{m}{(my^2-1)^{3/2}\sqrt{1-mr^2}} \tan^{-1}\left[\frac{r\sqrt{my^2-1}}{y\sqrt{1-mr^2}}\right]$$

$$- \frac{r}{y(y^2-r^2)(my^2-1)} \tag{3.6.22}$$

The second equation is obtained from the condition that $\sigma=0$ for $\rho>a$. We write from (Fabrikant, 1989a)

$$\sigma_0(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{x\,\mathrm{d}x}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{1}{x^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}(\rho_0)v(\rho_0,\phi),$$

$$\text{for}\quad \rho > a. \tag{3.6.23}$$

By using (11), (14), and (23) the following equation can be obtained:

$$-\frac{1}{\pi^2\rho}\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)f_2(\rho_0,\phi) + \frac{1}{\pi^2}\int_0^b \frac{\rho_0\,\mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{3/2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)f_1(\rho_0,\phi)$$

$$-\frac{1}{\pi^2\rho}\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{x\,\mathrm{d}x}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{1}{x^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}(\rho_0)v(\rho_0,\phi) = 0. \tag{3.6.24}$$

Let us apply the operator

$$\int_r^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\mathcal{L}\left(\frac{r}{\rho}\right)$$

to both sides of (24). The result is

$$\frac{1}{2\pi}f_2(r,\phi) + \frac{1}{\pi^2}\int_r^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\int_0^b \frac{\rho_0\,\mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{3/2}}\mathcal{L}\left(\frac{\rho_0 r}{\rho^2}\right)f_1(\rho_0,\phi)$$

$$+\frac{1}{2\pi}\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{r^2-\rho_0^2}}\mathcal{L}(\rho_0)\,v(\rho_0,\phi) = 0. \tag{3.6.25}$$

Again we can interchange the order of integration in (25) and perform the integration with respect to ρ, with the result

$$f_2(\rho,\phi) + \frac{1}{\pi^2}\int_0^{2\pi}\int_0^b Q_2(\rho,\rho_0,\phi-\phi_0)f_1(\rho_0,\phi_0)\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = g_2(\rho,\phi). \tag{3.6.26}$$

Here

$$g_2(\rho,\phi) = \rho\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{3/2}}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)v(\rho_0,\phi), \tag{3.6.27}$$

$$Q_2(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{\frac{\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}}}{(\rho e^{i(\phi-\phi_0)}-\rho_0)R}\tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\rho_0/\rho)}{(\rho/\rho_0)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\}-\frac{\rho_0}{R^2}, \tag{3.6.28}$$

and R is defined by (21). The following integral was used:

$$\int_r^\infty \frac{\rho^3\mathrm{d}\rho}{\sqrt{\rho^2-r^2}\,(\rho^2-y^2)^{3/2}(\rho^2-m)} = \frac{m}{(m-y^2)^{3/2}\sqrt{r^2-m}}\tan^{-1}\left[\frac{m-y^2}{r^2-m}\right]^{1/2}$$

$$-\frac{y^2}{(r^2-y^2)(m-y^2)}. \tag{3.6.29}$$

We note that $Q_1(\rho,\rho_0,\phi-\phi_0) = Q_2(\rho_0,\rho,\phi-\phi_0)$. This circumstance will allow us to decouple the equations by introduction of new variables. Indeed, substituting in (18) $t\sqrt{ab}$ instead of ρ and $\sqrt{ab}/x$ instead of ρ_0 yields

$$F_1(t,\phi)+\int_0^{2\pi}\int_0^k K(tx,\phi-\phi_0)\,F_2(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_1(t,\phi). \tag{3.6.30}$$

Here

$$F_1(t,\phi)=f_1(t\sqrt{ab},\phi),\quad F_2(t,\phi)=f_2(\sqrt{ab}/t,\phi)/t,\quad k=\sqrt{b/a}; \tag{3.6.31}$$

$$G_1(t,\phi)=g_1(t\sqrt{ab},\phi),\quad K(xt,\phi-\phi_0)=\frac{2}{\pi^2}\Re\left\{\frac{\gamma}{R_{xt}}\tan^{-1}\left(\frac{xt}{\gamma R_{xt}}\right)\right\}-\frac{xt}{\pi^2 R_{xt}^2}, \tag{3.6.32}$$

$$R_{xt}=\sqrt{1+x^2t^2-2xt\cos(\phi-\phi_0)},\quad \gamma=\frac{\sqrt{xte^{i(\phi-\phi_0)}}}{e^{i(\phi-\phi_0)}-xt}. \tag{3.6.33}$$

Equation (26) can be transformed by substitution $\rho=\sqrt{ab}/t$ and $\rho_0=x\sqrt{ab}$ to exactly the same form as (30), namely,

$$F_2(t,\phi) + \int_0^{2\pi}\int_0^k K(tx, \phi - \phi_0)\, F_1(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_2(t,\phi). \tag{3.6.34}$$

Here

$$G_2(t,\phi) = g_2(\sqrt{ab}/t)/t, \tag{3.6.35}$$

and all the remaining notations are given by (31)–(33). Equations (30) and (34) can be easily uncoupled by summation and subtraction

$$F_+(t,\phi) + \int_0^{2\pi}\int_0^k K(tx, \phi - \phi_0)\, F_+(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_+(t,\phi), \tag{3.6.36}$$

$$F_-(t,\phi) - \int_0^{2\pi}\int_0^k K(tx, \phi - \phi_0)\, F_-(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_-(t,\phi), \tag{3.6.37}$$

where

$$F_\pm = F_1 \pm F_2, \quad G_\pm = G_1 \pm G_2. \tag{3.6.38}$$

Thus the problem has been reduced to two independent integral equations (36) and (37) with elementary non-singular kernels which can be solved by iteration. Convergence of the iteration procedure is not guaranteed for k very close to unity which corresponds to the case of a very narrow annulus. Direct computation of the norm of the kernel in space L_2 gave the result of 0.41 for $k = 0.9$, and it was less than 0.8 for $k = 0.95$. It is then recommended to use an asymptotical solution for $k > 0.95$. We note that the arguments of the kernel x and t do not enter it independently but only as a product xt. The following integral representation is useful for computation of various integrals of the kernel:

$$K(y,\psi) = \frac{y}{2\pi^2}\int_0^1 \frac{\lambda(yz,\psi)\,\mathrm{d}z}{\sqrt{1-z}\,(1-y^2 z)^{3/2}}. \tag{3.6.39}$$

We recall that λ is defined by (6). Expression (39) shows that the kernel for each particular harmonic will also be an elementary function. For example, the kernels for the zero and first harmonic will be respectively

$$K_0(xt) = \frac{2}{\pi}\frac{xt}{1 - x^2 t^2}, \tag{3.6.40}$$

$$K_1(xt) = \frac{2}{\pi}\left[\frac{1}{1-x^2t^2} - \frac{1}{2xt}\ln\left(\frac{1+xt}{1-xt}\right)\right]. \tag{3.6.41}$$

Expression (40) is in agreement with the result of Clement and Love (1974). Expression (41) does not seem to have been reported in the literature. It is important to notice that various integral characteristics of interest can be expressed directly through the function f_2 (or F_2). For example, the total charge Q can be written as a limit

$$Q = \lim_{\rho\to\infty}\{\rho V_2(\rho,\phi,z)\}. \tag{3.6.42}$$

Substitution of (3) in (42) leads to

$$Q = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} f_2(\rho,\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi = \frac{\sqrt{ab}}{\pi^2}\int_0^{2\pi}\int_0^{k} F_2(x,\phi)\,\frac{\mathrm{d}x}{x}\,\mathrm{d}\phi. \tag{3.6.43}$$

The quantities proportional to magnetic polarizability can be found from

$$\nu_x = \lim_{\substack{\rho\to\infty\\ \phi=0}}\left[\rho^2\frac{\partial V_2}{\partial\phi}\right], \tag{3.6.44}$$

$$\nu_y = \lim_{\substack{\rho\to\infty\\ \phi=\pi/2}}\left[\rho^2\frac{\partial V_2}{\partial\phi}\right]. \tag{3.6.45}$$

Substitution of (3) in (44) and (45) yields respectively

$$\nu_x = \frac{2}{\pi^2}\int_0^{2\pi}\int_a^{\infty} f_2(\rho,\phi)\sin\phi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = \frac{2}{\pi^2}ab\int_0^{2\pi}\int_0^{k} F_2(x,\phi)\sin\phi\,\frac{\mathrm{d}x}{x^2}\,\mathrm{d}\phi, \tag{3.6.46}$$

$$\nu_y = -\frac{2}{\pi^2}\int_0^{2\pi}\int_a^{\infty} f_2(\rho,\phi)\cos\phi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = -\frac{2}{\pi^2}ab\int_0^{2\pi}\int_0^{k} F_2(x,\phi)\cos\phi\,\frac{\mathrm{d}x}{x^2}\,\mathrm{d}\phi. \tag{3.6.47}$$

Formulae (30)–(35), (43), and (46)–(47) are the main new results of this section.

Conducting annular disk charged to a unit potential. The governing integral equations in this case will take the form

$$F_1(t) + \int_0^k K_0(xt)\, F_2(x)\, \mathrm{d}x = G_1(t), \tag{3.6.48}$$

$$F_2(t) + \int_0^k K_0(xt)\, F_1(x)\, \mathrm{d}x = G_2(t), \tag{3.6.49}$$

where K_0 is defined by (40),$F_{1,2}$ and $G_{1,2}$ are understood as zero harmonics of the relevant notations (31),(32), and (35). In this particular case

$$G_1 = -\frac{t}{\sqrt{k^2 - t^2}} + \frac{kt}{\sqrt{1 - k^2 t^2}}, \tag{3.6.50}$$

$$G_2 = \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t\sqrt{1 - k^2 t^2}}. \tag{3.6.51}$$

Equations (48) and (49) were solved by Love (1976). We present here a slightly different version though based on the same idea, as well as a simple approximate treatment of the problem.

Assuming convergence of the iteration procedure, we can write the formal solution in the form

$$F_1 = \sum_{n=0}^{\infty} K_0^{2n}(G_1 - K_0 G_2), \tag{3.6.52}$$

$$F_2 = \sum_{n=0}^{\infty} K_0^{2n}(G_2 - K_0 G_1). \tag{3.6.53}$$

Here K_0^m is understood as the m-th iteration of the kernel. The first iteration in (53) ($n = 0$) yields

$$F_2^{(1)}(t) = \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t\sqrt{1 - k^2 t^2}} - \frac{2}{\pi}\, t \int_0^k \left[\frac{k}{\sqrt{1 - k^2 x^2}} - \frac{1}{\sqrt{k^2 - x^2}} \right] \frac{x^2\, \mathrm{d}x}{1 - x^2 t^2}$$

$$= \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t} + \theta(t), \tag{3.6.54}$$

where the notation was introduced

$$\theta(t) = \frac{2}{\pi t}\left[\sin^{-1}(k^2) - \frac{k}{\sqrt{k^2 - t^2}}\sin^{-1}\left(\frac{k\sqrt{k^2 - t^2}}{\sqrt{1 - k^2 t^2}}\right)\right]. \tag{3.6.55}$$

Now we need to consider the action of K_0^2 on the first iteration (54).

$$K_0^2 F_2^{(1)}(t) = \frac{4}{\pi^2}\int_0^k \frac{ts\,\mathrm{d}s}{1 - t^2 s^2}\int_0^k \left[\frac{k}{x\sqrt{k^2 - x^2}} - \frac{1}{x}\right]\frac{sx\,\mathrm{d}x}{1 - s^2 x^2} + K_0^2\theta(t)$$

$$= \frac{2}{\pi}kt\int_0^k \frac{s^2\,\mathrm{d}s}{(1 - t^2 s^2)\sqrt{1 - k^2 s^2}} - \frac{2}{\pi^2}\int_0^k \ln\left(\frac{1 + ks}{1 - ks}\right)\frac{ts\,\mathrm{d}s}{1 - t^2 s^2} + K_0^2\theta(t)$$

$$= -\theta(t) - \int_0^k K_0^2(xt)\,\frac{\mathrm{d}x}{x} + K_0^2\theta(t). \tag{3.6.56}$$

Substitution of (56) in (53) shows that all the expressions containing $\theta(t)$ will cancel out at each subsequent iteration. This allows us to write the exact solution in the form

$$F_2(t) = \frac{1}{t}\left[\frac{k}{\sqrt{k^2 - t^2}} - 1\right] - \sum_{n=1}^{\infty}\int_0^k K_0^{2n}(xt)\,\frac{\mathrm{d}x}{x}. \tag{3.6.57}$$

Since all iterated kernels are positive, the term in square brackets in (57) gives the upper bound for the solution. Substitution of (57) in (48) yields, after simplification,

$$F_1(t) = -\frac{t}{\sqrt{k^2 - t^2}} + \sum_{n=0}^{\infty}\int_0^k K_0^{2n+1}(xt)\,\frac{\mathrm{d}x}{x}. \tag{3.6.58}$$

Capacitance of the annulus can be found by substitution of (57) in (43), with the result

$$C = \frac{2}{\pi} a \left\{ 1 - k \sum_{n=1}^{\infty} \int_0^k \int_0^k K_0^{2n}(xt) \frac{\mathrm{d}x}{x} \frac{\mathrm{d}t}{t} \right\}. \tag{3.6.59}$$

Taking into consideration that

$$C_0 = \frac{2}{\pi} a \tag{3.6.60}$$

is the capacity of a circular disk of radius a, we can write the expression for the dimensionless capacity C^* which is defined as the ratio

$$C^* = \frac{C}{C_0} = 1 - k \sum_{n=1}^{\infty} \int_0^k \int_0^k K_0^{2n}(xt) \frac{\mathrm{d}x}{x} \frac{\mathrm{d}t}{t}. \tag{3.6.61}$$

The symmetry of x and t in (61) allows us to reduce the order of iterated kernel as follows:

$$C^* = 1 - k \sum_{n=1}^{\infty} \int_0^k \left[\int_0^k K_0^{n}(xt) \frac{\mathrm{d}x}{x} \right]^2 \mathrm{d}t. \tag{3.6.62}$$

Yet another approximate solution for the dimensionless capacity C^* can be found by a different method. Indeed, from (43) we have

$$C^* = k \int_0^k F_2(t) \frac{\mathrm{d}t}{t}. \tag{3.6.63}$$

Multiplying both sides of (49) by k/t and integrating with respect to t from 0 to k, we obtain

$$k \int_0^k F_2(t) \frac{\mathrm{d}t}{t} + \frac{k}{\pi} \int_0^k \ln\left(\frac{1 + kx}{1 - kx} \right) F_1(x)\, \mathrm{d}x = \sqrt{1 - k^4} \tag{3.6.64}$$

We can express F_1 from (48) and substitute it into (64). The result is

$$k\int_0^k F_2(t)\,\frac{\mathrm{d}t}{t} - \frac{2k}{\pi^2}\int_0^k T(x)\,F_2(x)\,\frac{\mathrm{d}x}{x} = \frac{2}{\pi}\left[\cos^{-1}(k^2) + \frac{1}{2}\sqrt{1-k^4}\,\ln\left(\frac{1+k^2}{1-k^2}\right)\right], \tag{3.6.65}$$

where

$$T(x) = x^2\int_0^k \ln\left(\frac{1+kt}{1-kt}\right)\frac{t\,\mathrm{d}t}{1-x^2t^2}. \tag{3.6.66}$$

Now we can use the mean value theorem to obtain

$$C^* = \frac{2}{\pi}\left[\cos^{-1}(k^2) + \frac{1}{2}\sqrt{1-k^4}\,\ln\left(\frac{1+k^2}{1-k^2}\right)\right]\left(1 - \frac{2}{\pi^2}\,T(X)\right)^{-1}. \tag{3.6.67}$$

According to the mean value theorem, we know about X only that it is located somewhere in the interval $[0,k]$. One needs to find an optimum value for X in order to make (67) useful. This exercise is beyond the scope of this book. When $X=0$, formula (67) coincides with the result of Smythe (1951) who obtained it from physical considerations. Since $T(X)$ is non-negative, the term in square brackets in (67) gives the lower bound. Note also that it is exact in two extreme cases, namely, $k=0$ and $k=1$.

Magnetic polarizability of a circular annulus. In this case we may assume, without loss of generality, that

$$v(\rho,\phi) = v_1\rho\cos\phi, \tag{3.6.68}$$

where v_1 is a constant. The governing integral equations will take the form

$$F_1(t) + \int_0^k K_1(xt)\,F_2(x)\,\mathrm{d}x = G_1(t), \tag{3.6.69}$$

$$F_2(t) + \int_0^k K_1(xt)\,F_1(x)\,\mathrm{d}x = G_2(t), \tag{3.6.70}$$

where K_1 is defined by (41),$F_{1,2}$ and $G_{1,2}$ are understood as first harmonics of the relevant notations (31),(32), and (35). In this particular case

$$G_1(t) = v_1\sqrt{ab}t^2\left[\frac{k}{\sqrt{1-k^2t^2}} - \frac{1}{\sqrt{k^2-t^2}}\right], \tag{3.6.71}$$

$$G_2(t) = v_1\frac{\sqrt{ab}}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - \frac{2-k^2t^2}{\sqrt{1-k^2t^2}}\right]. \tag{3.6.72}$$

Equations (69) and (70) have not been considered before. We employ the same method as above. Assuming convergence of the iteration procedure, we can write the formal solution in the form

$$F_1 = \sum_{n=0}^{\infty} K_1^{2n}(G_1 - K_1G_2), \tag{3.6.73}$$

$$F_2 = \sum_{n=0}^{\infty} K_1^{2n}(G_2 - K_1G_1). \tag{3.6.74}$$

Here K_1^m is understood as the m-th iteration of the kernel. The first iteration in (74) yields

$$F_2^{(1)}(\mathrm{t}) = v_1\frac{\sqrt{ab}}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - 2\right] + \theta_1(t). \tag{3.6.75}$$

Here the notation was introduced

$$\theta_1(t) = \frac{2v_1\sqrt{ab}}{\pi t^2}\left[2\sin^{-1}(k^2) - \frac{t}{2k}\sqrt{1-k^4}\ \ln\left(\frac{1+kt}{1-kt}\right)\right.$$
$$\left. - \frac{2k^2-t^2}{k\sqrt{k^2-t^2}}\sin^{-1}\left(\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right)\right]. \tag{3.6.76}$$

Now we need to compute

$$K_1^2F_2^{(1)}(t) = \int_0^k K_1(ts)\,\mathrm{d}s\int_0^k K_1(sx)\frac{v_1\sqrt{ab}}{x^2}\left[\frac{2k^2-x^2}{k\sqrt{k^2-x^2}} - 2\right]\mathrm{d}x + K_1^2\theta_1(t). \tag{3.6.77}$$

Integration with respect to x in (77) yields

$$K_1^2 F_2^{(1)}(t) = \frac{2}{\pi} v_1 \sqrt{ab} \int_0^k \left[\frac{1}{k} + \frac{\pi}{2} \frac{ks^2}{\sqrt{1-k^2s^2}} - \frac{1+k^2s^2}{2k^2s} \ln\left(\frac{1+ks}{1-ks}\right)\right] K_1(ts)\,\mathrm{d}s + K_1^2\theta_1(t). \tag{3.6.78}$$

One can easily verify that

$$\int_0^k \frac{ks^2}{\sqrt{1-k^2s^2}} K_1(st)\,\mathrm{d}s = -\frac{\theta_1(t)}{v_1\sqrt{ab}}. \tag{3.6.79}$$

Substitution of (79) in (78) gives

$$K_1^2 F_2^{(1)}(t) = -\theta_1(t) + \frac{2}{\pi} v_1 \sqrt{ab} \int_0^k \left[\frac{1}{k} - \frac{1+k^2s^2}{2k^2s} \ln\left(\frac{1+ks}{1-ks}\right)\right] K_1(st)\,\mathrm{d}s + K_1^2\theta_1(t). \tag{3.6.80}$$

By using the identity

$$\frac{1}{k} - \frac{1+k^2s^2}{2k^2s} \ln\left(\frac{1+ks}{1-ks}\right) = -\pi \int_0^k K_1(xs)\,\frac{\mathrm{d}x}{x^2}, \tag{3.6.81}$$

we can further simplify (80), namely,

$$K_1^2 F_2^{(1)}(t) = -\theta_1(t) - 2v_1\sqrt{ab} \int_0^k K_1^2(st)\,\frac{\mathrm{d}s}{s^2} + K_1^2\theta_1(t). \tag{3.6.82}$$

Finally, substitution of (82) in (74) gives the solution

$$F_2(t) = v_1\sqrt{ab}\left\{ \frac{1}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - 2\right] - 2\sum_{n=1}^{\infty} \int_0^k K_1^{2n}(st)\,\frac{\mathrm{d}s}{s^2}\right\}. \tag{3.6.83}$$

Since iterated kernels are all positive, the first term in (83) gives the upper bound for F_2. Substitution of (83) in (69) allows us to find

$$F_1(t) = v_1\sqrt{ab}\left\{-\frac{t^2}{\sqrt{k^2-t^2}} + 2\sum_{n=0}^{\infty}\int_0^k K_1^{2n+1}(st)\,\frac{\mathrm{d}s}{s^2}\right\}. \tag{3.6.84}$$

Formulae (83) and (84) give the complete solution to the problem. We introduce the dimensionless magnetic polarizability ν^* as the ratio of magnetic polarizability of the annulus ν_y to that of a circular disk of radius a $\nu_0 = 4v_1a^3/(3\pi)$. Substitution of (83) in (47) yields, after integration,

$$\nu^* = 1 - 3k^3\sum_{n=1}^{\infty}\int_0^k\int_0^k K_1^{2n}(st)\,\frac{\mathrm{d}s\,\mathrm{d}t}{s^2t^2}. \tag{3.6.85}$$

Again, the symmetry of (85) allows us to reduce the order of the kernel iteration by writing

$$\nu^* = 1 - 3k^3\sum_{n=1}^{\infty}\int_0^k\left[\int_0^k K_1^{n}(st)\,\frac{\mathrm{d}s}{s^2}\right]^2\mathrm{d}t. \tag{3.6.86}$$

Since all iterated kernels are positive, truncation in (86) gives the upper bound for ν^*. A simple approximate formula for small k can be derived by integration of (75), with the result

$$\nu^* = 1 - \frac{2}{\pi}\left\{\sin^{-1}(k^2) + \frac{\sqrt{1-k^4}}{4}\left[k^2 - \frac{5+k^4}{2}\ln\left(\frac{1+k^2}{1-k^2}\right)\right]\right\}. \tag{3.6.87}$$

Formula (87) is exact in two extreme cases, namely, for $k=0$ and for $k=1$. The series expansion of (87) gives

$$\nu^* = 1 - \frac{2}{\pi}\left(\frac{k^{10}}{5} + \frac{9k^{14}}{70} + \frac{233k^{18}}{2520}\right) + \mathrm{O}(k^{22}). \tag{3.6.88}$$

It is of interest to notice that all powers of k below 10 cancelled out as compared to the relevant expression for capacity where the series expansion starts with the sixth power of k. This means that ν^* will be not far away from unity even for not so small k. For example, ν^* is greater than 0.98 for $k=0.8$. Convergence of the iteration procedure was investigated by computing the norm of K_1. It was found to be much less than 1 up to the ratio $b/a = 0.9$. Even for $b/a = 0.999$ the norm is equal to 0.6 which still assures a good convergence.

3.7. Neumann problem for a circular annulus

The problem is reduced to a two-dimensional integral equation with an elementary non-singular kernel. Several specific examples are considered. Exact solution has been obtained in terms of the iterated kernel.

A number of papers have been published on the Neumann potential problem for a circular annulus. One of the first significant works on the subject was that of Collins (1963) who considered the axisymmetric problem by means of integral equations. A new and interesting approach was developed by Clements and Love (1974) where the reader can also find some additional references. Quite a few papers are devoted to the mathematically equivalent problem of an annular crack in an elastic space. One can find the relevant references in Clements and Ang (1988). All the works published are devoted to the axisymmetric problems only. No solution to the general problem has been attempted as yet. This kind of solution is now possible due to the results described in Chapter 1. The problem is reduced to a set of two two-dimensional Fredholm integral equations with an elementary non-singular kernel which can be solved by iterations. This set can be easily uncoupled. The case of a circular annulus, with a prescribed charge density distribution, and the rest of the plane being grounded, is considered in detail. The governing integral equations are solved exactly in series involving the iterated kernels.

Theory. We have a mixed boundary value problem of potential theory for a half space $z \geq 0$. We need to find a harmonic function V vanishing at infinity and satisfying the following conditions at $z = 0$:

$$-\frac{1}{2\pi}\frac{\partial V}{\partial z} \equiv \sigma(\rho,\phi) = p(\rho,\phi), \quad \text{for} \quad b < \rho < a, \quad 0 \leq \phi < 2\pi;$$

$$V = 0, \quad \text{for} \quad \rho < b \quad \text{or} \quad \rho > a, \quad 0 \leq \phi < 2\pi. \tag{3.7.1}$$

Here p is a known function. The approach proposed here is similar to the one used in previous section. Let us introduce two harmonic functions

$$V_1(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_0^b \frac{\sqrt{l_2^2(\rho_0) - \rho_0^2}\; f_1(\rho_0,\phi_0)\, d\rho_0\, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2}, \tag{3.7.2}$$

$$V_2(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_a^\infty \frac{\sqrt{\rho_0^2 - l_1^2(\rho_0)}\; f_2(\rho_0,\phi_0)\, d\rho_0\, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2}. \tag{3.7.3}$$

Here the notation was introduced

$$l_1(x) = \frac{1}{2}\{\sqrt{(\rho + x)^2 + z^2} - \sqrt{(\rho - x)^2 + z^2}\},$$

$$l_2(x) = \frac{1}{2}\{\sqrt{(\rho + x)^2 + z^2} + \sqrt{(\rho - x)^2 + z^2}\},$$

One can verify that the potential V_1 is due to some charge distribution σ_1 on the interval $\rho \le b$, and zero charge distribution outside this interval. The potential V_2 is due to σ_2 as the charge density distribution for $\rho \ge a$, and zero charge elsewhere. The functions f_1 and f_2 can be related to the relevant charge density distributions from the following considerations. On the plane $z = 0$ formula (2) gives

$$V_1(\rho, \phi, 0) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\min(\rho,b)} \frac{\sqrt{\rho^2 - \rho_0^2}\, f_1(\rho_0, \phi_0)\, d\rho_0\, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)} = 4\int_0^{\min(\rho,b)} \frac{dx}{\sqrt{\rho^2 - x^2}}\, \mathcal{L}\left(\frac{x}{\rho}\right) f_1(x, \phi). \tag{3.7.4}$$

The utility of the formal change of the notation ρ_0 to x will be clear from the comparison with (5). As before, the $\mathcal{L}$-operator is defined as

$$\mathcal{L}(k) f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi - \phi_0)\, f(\phi_0)\, d\phi_0,$$

with

$$\lambda(k, \psi) = \frac{1 - k^2}{1 + k^2 - 2k\cos\psi}.$$

On the other hand, we have from (Fabrikant, 1989a) the following expression for the potential in the plane $z = 0$ due to an arbitrary charge distribution σ_1 inside a circle $\rho = b$:

$$V_1(\rho, \phi, 0) = 4\int_0^{\min(\rho,b)} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^b \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}}\, \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma_1(\rho_0, \phi). \tag{3.7.5}$$

Comparison of (4) and (5) immediately gives

$$f_1(x,\phi)=\int_x^b \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma_1(\rho_0,\phi). \tag{3.7.6}$$

The inverse of (6) is readily available, and is

$$\sigma_1(\rho,\phi)=-\frac{2}{\pi}\frac{\mathcal{L}(\rho)}{\rho}\frac{d}{d\rho}\int_\rho^b \frac{x\,dx}{\sqrt{x^2-\rho^2}}\,\mathcal{L}\left(\frac{1}{x}\right)f(x,\phi). \tag{3.7.7}$$

On the plane $z=0$ formula (3) gives

$$V_2(\rho,\phi,0)=\frac{2}{\pi}\int_0^{2\pi}\int_{\max(a,\rho)}^{\infty}\frac{\sqrt{\rho_0^2-\rho^2}\;f_2(\rho_0\phi_0)\,d\rho_0 d\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}=4\int_{\max(a,\rho)}^{\infty}\frac{dx}{\sqrt{x^2-\rho^2}}\,\mathcal{L}\left(\frac{\rho}{x}\right)f_2(x,\phi). \tag{3.7.8}$$

In a similar manner, we can find from (Fabrikant, 1989a) that the potential on the plane $z=0$ due to the charge distribution σ_2 which is non-zero outside the circle $\rho=a$ and zero elsewhere, is given by

$$V_2(\rho,\phi,0)=4\int_{\max(\rho,a)}^{\infty}\frac{dx}{\sqrt{x^2-\rho^2}}\int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_2(\rho_0,\phi). \tag{3.7.9}$$

Again, comparison of (8) and (9) yields

$$f_2(x,\phi)=\int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma(\rho_0,\phi). \tag{3.7.10}$$

The inversion of (10) yields

$$\sigma_2(\rho,\phi)=\frac{2}{\pi\rho}\,\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_a^\rho \frac{\rho_0 d\rho_0}{\sqrt{\rho^2-\rho_0^2}}\,\mathcal{L}(\rho_0)\,f_2(\rho_0,\phi). \tag{3.7.11}$$

Functions f_1 and f_2 in (2) and (3) can be found from the second condition (1). We can assume now that the charge distribution is given all over the plane $z=0$, namely, it is assumed to be σ_1 on the interval $[0,b]$, it is equal to p on the annulus $b\le\rho\le a$, and it is equal to σ_2 for $\rho>a$. In this case the relevant potential can be related to the charge distribution in two different ways, namely, (Fabrikant, 1989a):

$$V(\rho,\phi) = 4\int_0^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_x^{\infty} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \tag{3.7.12}$$

$$V(\rho,\phi) = 4\int_{\rho}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^2 - \rho^2}} \int_0^{x} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{3.7.13}$$

Substitution of the second boundary condition (1) in (13) for $\rho > a$ yields

$$0 = 4\int_{\rho}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^2 - \rho^2}} \left\{ \int_0^{b} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_1(\rho_0,\phi) + \int_b^{a} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)p(\rho_0,\phi) \right.$$

$$\left. + \int_a^{x} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_2(\rho_0,\phi) \right\},$$

which immediately leads to

$$\int_0^{b} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_1(\rho_0,\phi) + \int_b^{a} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)p(\rho_0,\phi)$$

$$+ \int_a^{x} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_2(\rho_0,\phi) = 0. \tag{3.7.14}$$

Substitution of (7) and (10) in (14) gives the first equation relating f_1 and f_2

$$f_2(x,\phi) - \frac{2}{\pi}\int_0^{b} \frac{\mathrm{d}\rho}{\sqrt{x^2 - \rho^2}} \mathcal{L}\left(\frac{\rho^2}{x}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{b} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - \rho^2}} \mathcal{L}\left(\frac{1}{\rho_0}\right) f_1(\rho_0,\phi)$$

$$= -\int_b^{a} \frac{\rho \mathrm{d}\rho}{\sqrt{x^2 - \rho^2}} \mathcal{L}\left(\frac{\rho}{x}\right)p(\rho,\phi). \tag{3.7.15}$$

We can interchange the order of integration in the second term of (15) according to the rule

$$\int_0^b F(r)\,\mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{f(\rho)\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}} = -\int_0^b f(\rho)\,\mathrm{d}\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^b \frac{F(r)\,r\mathrm{d}r}{\sqrt{r^2-\rho^2}}, \tag{3.7.16}$$

and to integrate with respect to ρ. The result can be simplified as follows:

$$f_2(\rho,\phi) + \int_0^{2\pi}\int_0^b K_1(\rho,\rho_0,\phi-\phi_0)\,f_1(\rho_0,\phi_0)\,\mathrm{d}\rho_0\mathrm{d}\phi_0 = g_2(\rho,\phi), \tag{3.7.17}$$

where

$$K_1(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{\frac{\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}}}{\pi^2 R\,(\rho-\rho_0 e^{i(\phi-\phi_0)})}\tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\rho_0/\rho)}{(\rho/\rho_0)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\} + \frac{\rho}{\pi^2 R^2},$$

$$R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0), \quad g_2(\rho,\phi) = -\int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho^2-\rho_0^2}}\,\mathcal{L}\!\left(\frac{\rho_0}{\rho}\right)p(\rho_0,\phi). \tag{3.7.18}$$

Here we used the following result:

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}\sqrt{x^2-\rho^2}\,(1-m\rho^2)}$$

$$= \frac{mr}{\sqrt{mx^2-1}(1-mr^2)^{3/2}}\tan^{-1}\frac{r\sqrt{mx^2-1}}{x\sqrt{1-mr^2}} + \frac{x}{(x^2-r^2)\,(1-mr^2)}. \tag{3.7.19}$$

In a similar manner, we get from (12) for $\rho<b$

$$\int_\rho^b \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\!\left(\frac{\rho}{\rho_0}\right)\sigma_1(\rho_0,\phi) + \int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\!\left(\frac{\rho}{\rho_0}\right)p(\rho_0,\phi)$$

$$+\int_a^\infty \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\!\left(\frac{\rho}{\rho_0}\right)\sigma_2(\rho_0,\phi) = 0. \tag{3.7.20}$$

Substitution of (6) and (11) in (20) yields, after interchanging the order of integration and subsequent simplification to

$$f_1(\rho,\phi) + \int_0^{2\pi}\int_a^{\infty} K_2(\rho,\rho_0,\phi-\phi_0)\, f_2(\rho_0,\phi_0)\, d\rho_0 d\phi_0 = g_1(\rho,\phi), \tag{3.7.21}$$

where

$$K_2(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{\frac{\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}}}{\pi^2 R\,(\rho_0-\rho e^{i(\phi-\phi_0)})}\tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\rho/\rho_0)}{(\rho_0/\rho)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\} + \frac{\rho_0}{\pi^2 R^2},$$

$$R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0), \quad g_1(\rho,\phi) = -\int_b^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right) p(\rho_0,\phi). \tag{3.7.22}$$

Here we used the following result:

$$\int_a^{\infty} F(\rho)\, d\rho \frac{d}{d\rho}\int_{\rho}^{\infty}\frac{f(r)\, r dr}{\sqrt{r^2-\rho^2}} = -\int_a^{\infty} f(r)\, dr \frac{d}{dr}\int_a^{r}\frac{F(\rho)\,\rho d\rho}{\sqrt{r^2-\rho^2}} \tag{3.7.23}$$

in order to interchange the order of integration. The integration itself was performed by using

$$\frac{d}{dr}\int_r^{\infty}\frac{\rho^3 d\rho}{\sqrt{\rho^2-x^2}\sqrt{\rho^2-r^2}\,(\rho^2-m)}$$

$$= -\frac{mr}{\sqrt{m-x^2}\,(r^2-m)^{3/2}}\tan^{-1}\left[\frac{m-x^2}{r^2-m}\right]^{1/2} - \frac{mr}{(r^2-m)\,(r^2-x^2)}. \tag{3.7.24}$$

Though the kernels of integral equations (17) and (21) look somewhat different, a simple change of variables can make them identical. Indeed, introduce new variables x and t which are related to the old ones by the relationships $\rho = \sqrt{ab}/t$ and $\rho_0 = \sqrt{ab}\,x$. These substitutions are to be used in (17). The slightly different relationships, namely, $\rho = \sqrt{ab}\,t$ and $\rho_0 = \sqrt{ab}/x$ are to be substituted in (21). The final result is

$$F_1(t,\phi) + \int_0^{2\pi}\int_0^k K(xt,\phi-\phi_0)\, F_2(x,\phi_0)\, dx\, d\phi_0 = G_1(t,\phi),$$

$$F_2(t,\phi) + \int_0^{2\pi}\int_0^k K(xt,\phi-\phi_0)\, F_1(x,\phi_0)\, dx\, d\phi_0 = G_2(t,\phi), \qquad (3.7.25)$$

where

$$F_1(t,\phi) = f_1(\sqrt{ab}\, t,\phi), \quad F_2(t,\phi) = f_2(\sqrt{ab}/t,\phi)/t,$$

$$G_1(t,\phi) = g_1(\sqrt{ab}\, t,\phi), \quad G_2(t,\phi) = g_2(\sqrt{ab}/t,\phi)/t, \qquad (3.7.26)$$

$$K(xt,\phi-\phi_0) = \frac{1}{\pi^2}\left\{ 2\Re\left[\frac{\bar{\gamma}}{R_{xt}}\tan^{-1}\left(\frac{xt}{\gamma R_{xt}}\right)\right] + \frac{1}{R_{xt}^2}\right\}, \qquad (3.7.27)$$

$$k = \sqrt{b/a}, \quad \gamma = \frac{\sqrt{xt e^{i(\phi-\phi_0)}}}{e^{i(\phi-\phi_0)} - xt}, \quad R_{xt} = \sqrt{1 + x^2t^2 - 2\,xt\cos(\phi-\phi_0)}. \qquad (3.7.28)$$

and the overbar denotes the complex conjugate value. Note also the relationship $xt/(\gamma R_{xt}) = R_{xt}\bar{\gamma}$. Since integral equations (25) have the same kernel, they can be uncoupled by simple addition and subtraction. The result can be presented as

$$F_+(t,\phi) + \int_0^{2\pi}\int_0^k K(xt,\phi-\phi_0)\, F_+(x,\phi_0)\, dx\, d\phi_0 = G_+(t,\phi),$$

$$F_-(t,\phi) - \int_0^{2\pi}\int_0^k K(xt,\phi-\phi_0)\, F_-(x,\phi_0)\, dx\, d\phi_0 = G_-(t,\phi), \qquad (3.7.29)$$

with

$$F_\pm = F_1 \pm F_2, \quad G_\pm = G_1 \pm G_2. \qquad (3.7.30)$$

Formulae (25)–(30) are the main new results of this section. Some quantities which are of interest in applications can be expressed directly through functions F_1 and F_2. For example, in elastic crack problems the stress intensity factors are given by

$$K_a(\phi) = \frac{2kF_2(k,\phi)}{\pi\sqrt{2a}}, \quad K_b(\phi) = \frac{2F_1(k,\phi)}{\pi\sqrt{2b}}.$$

In electromagnetic problems, the total charge inside the circle $\rho = b$ is

$$Q_1 = \frac{2}{\pi}\sqrt{ab} \int_0^{2\pi}\int_0^k F_1(t,\phi)\,\mathrm{d}t\,\mathrm{d}\phi$$

The similar quantity outside the circle $\rho = a$ is

$$Q_2 = \sqrt{ab}\int_0^{2\pi} F_2(0,\phi)\,\mathrm{d}\phi.$$

In the annular crack problems quantities Q_1 and Q_2 correspond to the relevant resultant forces.

Uniform charge distribution. Let us consider in more detail the case of $p = p_0 = \mathrm{const.}$ The governing integral equations in this case will take the form

$$F_1(t) + \frac{2}{\pi}\int_0^k \frac{F_2(x)\,\mathrm{d}x}{1 - x^2t^2} = p_0\sqrt{ab}\left[\sqrt{k^2 - t^2} - \frac{\sqrt{1 - k^2t^2}}{k}\right],$$

$$F_2(t) + \frac{2}{\pi}\int_0^k \frac{F_1(x)\,\mathrm{d}x}{1 - x^2t^2} = p_0\frac{\sqrt{ab}}{t^2}\left[\frac{\sqrt{k^2 - t^2}}{k} - \sqrt{1 - k^2t^2}\right]. \tag{3.7.31}$$

Denote the kernel of (31) as

$$K_0(xt) = \frac{2}{\pi}\frac{1}{1 - x^2t^2}.$$

Equations (31) are in agreement with the results of Clements and Love (1974). Their numerical solution was given in Clements and Ang (1988). We present here an analytical solution. Let us use the method of iteration. Assuming the right hand sides in (31) as zero approximations, we can write for the first approximation

$$F_2^{(1)} = p_0\sqrt{ab}\left\{\frac{1}{t^2}\left[\frac{\sqrt{k^2-t^2}}{k} - \sqrt{1-k^2t^2}\right] - \frac{2}{\pi}\int_0^k \frac{\sqrt{k^2-x^2} - \sqrt{(1/k^2)-x^2}}{1-x^2t^2}\,\mathrm{d}x\right\}. \tag{3.7.32}$$

Simplifications in (32) are elementary, and the final result is

$$F_2^{(1)}(t) = p_0\frac{\sqrt{ab}}{t^2}\left[\frac{\sqrt{k^2-t^2}}{k} - 1\right] + \theta(t), \tag{3.7.33}$$

with

$$\theta(t) = p_0\frac{2\sqrt{ab}}{\pi t^2}\left[\sin^{-1}(k^2) - \frac{\sqrt{k^2-t^2}}{k}\sin^{-1}\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right]. \tag{3.7.34}$$

Now we need to investigate the action of the second iterated kernel K_0^2 on (33). The relevant expression is

$$K_0^2F_2^{(1)}(t) = \frac{4}{\pi^2}p_0\sqrt{ab}\int_0^k \frac{\mathrm{d}x}{1-x^2t^2}\int_0^k \frac{\mathrm{d}s}{1-s^2x^2}\frac{1}{s^2}\left[\frac{\sqrt{k^2-s^2}}{k} - 1\right] + K_0^2\theta(t). \tag{3.7.35}$$

Taking into consideration that

$$\theta(t) = \frac{1}{k}\int_0^k \sqrt{1-k^2x^2}\,K_0(xt)\,\mathrm{d}x,$$

expression (35) can be presented as

$$K_0^2F_2^{(1)}(t) = -\theta(t) + \frac{2}{\pi}p_0\sqrt{ab}\int_0^k \psi(x)\,K_0(xt)\,\mathrm{d}x + K_0^2\theta(t), \tag{3.7.36}$$

with

$$\psi(x) = \frac{1}{k} - \frac{x}{2}\ln\left(\frac{1+kx}{1-kx}\right). \tag{3.7.37}$$

Since the exact solution for F_2 can be represented as

$$F_2(t) = F_2^{(1)}(t) + \sum_{n=1}^{\infty} \int_0^k F_2^{(1)}(x)\, K_0^{2n}(x,t)\, \mathrm{d}x, \tag{3.7.38}$$

we can see that each next iteration will cancel expression containing θ from the previous iteration, so that θ would not enter the final solution which now takes the form

$$F_2(t) = p_0\sqrt{ab}\left\{\frac{1}{t^2}\left[\frac{\sqrt{k^2-t^2}}{k} - 1\right] + \frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^k \psi(x)\, K_0^{2n+1}(x,t)\, \mathrm{d}x\right\}. \tag{3.7.39}$$

Substitution of (39) in the first equation of (31) yields

$$F_1(t) = p_0\sqrt{ab}\left\{\sqrt{k^2-t^2} - \frac{2}{\pi}\left[\psi(t) + \sum_{n=1}^{\infty}\int_0^k \psi(x)\, K_0^{2n}(x,t)\, \mathrm{d}x\right]\right\}. \tag{3.7.40}$$

Formulae (39) and (40) give an exact solution of (31) in terms of the iterated kernel. Actual computations were made for the set of values of $\{b/a\} := 0.04, 0.1, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99$. The dimensionless quantities $F_1^* = F_1/(p_0\sqrt{ab}\,)$ and $F_2^* = F_2/(p_0\sqrt{ab}\,)$ are presented in Fig. 3.7.1 and Fig. 3.7.2 respectively versus $\xi = 1 + 300(t/k)$. This choice allowed us to plot all curves on the same base. In order to avoid overlapping, not all curves were actually plotted. A comparison was also made with the numerical data for the stress intensity factors presented by Clements and Ang. The agreement was found to be excellent.

The case of non-axisymmetric charge distribution prescribed over the annulus can be treated in a similar manner. For example, let

$$p(\rho,\phi) = p_1\rho\cos\phi. \tag{3.7.41}$$

The governing integral equations in this case will take the form

$$F_1(t) + \frac{2}{\pi}\int_0^k \left[\frac{xt}{1-x^2t^2} + \frac{1}{2}\ln\left(\frac{1+xt}{1-xt}\right)\right] F_2(x)\, \mathrm{d}x = p_1abt\left[\sqrt{k^2-t^2} - \frac{\sqrt{1-k^2t^2}}{k}\right],$$

Fig. 3.7.1. Uniform charge distribution (solution for F_1)

Fig. 3.7.2. Uniform charge distribution (solution for F_2)

$$F_2(t)+\frac{2}{\pi}\int_0^k\left[\frac{xt}{1-x^2t^2}+\frac{1}{2}\ln\left(\frac{1+xt}{1-xt}\right)\right]F_1(x)\,\mathrm{d}x=\frac{p_1ab}{3t^2}\left[\frac{2k^2+t^2}{k^3}\sqrt{k^2-t^2}\right.$$

$$\left.-(2+k^2t^2)\sqrt{1-k^2t^2}\right]. \tag{3.7.42}$$

Equations (42) can be solved by any standard numerical procedure.

3.8. Alternative approach to the Dirichlet problem

The solution to the Dirichlet problem for a circular annulus, presented in section 3.6 is not the only one to use. Several alternatives may be suggested. The boundary conditions, as before, are

$$V(\rho,\phi,0)=v(\rho,\phi),\quad \text{for}\quad b<\rho<a,\quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for}\quad \rho<b\quad \text{or}\quad \rho>a,\quad 0\le\phi<2\pi. \tag{3.8.1}$$

The governing integral equation will take the form

$$\int_0^{2\pi}\int_b^a\frac{\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}=v(\rho,\phi). \tag{3.8.2}$$

By using the integral representation

$$\frac{1}{R}=\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}=\frac{2}{\pi}\int_{max(\rho_0,\rho)}^{\infty}\frac{\lambda(\frac{\rho\rho_0}{x^2},\phi-\phi_0)\,\mathrm{d}x}{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}}, \tag{3.8.3}$$

The governing integral equation may be reduced to

$$4\int_\rho^a\frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_b^x\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$+4\int_a^\infty \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)=v(\rho,\phi).$$

(3.8.4)

Application of the operator

$$\mathcal{L}(r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)$$

to both sides of (4) yields

$$-2\pi\int_b^r \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{r^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho_0}{r}\right)\sigma(\rho_0,\phi)-\frac{4r}{\sqrt{a^2-r^2}}\int_a^\infty \frac{\sqrt{x^2-a^2}\,\mathrm{d}x}{x^2-r^2}\int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\left(\frac{r\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$=\mathcal{L}(r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)v(\rho,\phi). \tag{3.8.5}$$

We introduce a new unknown function

$$\psi(r,\phi)=\int_b^r \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{r^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho_0}{r}\right)\sigma(\rho_0,\phi). \tag{3.8.6}$$

The inverse of (6) is readily available, and is

$$\sigma(\rho,\phi)=\frac{2}{\pi\rho}\,\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_b^\rho \frac{r\,\mathrm{d}r}{\sqrt{\rho^2-r^2}}\,\mathcal{L}(r)\,\psi(r,\phi). \tag{3.8.7}$$

Substitution of (6) in (5) gives

$$-2\pi\psi(r,\phi)-\frac{8}{\pi}\frac{r}{\sqrt{a^2-r^2}}\int_a^\infty \frac{x^2-a^2}{x^2-r^2}\,\mathrm{d}x\int_b^a \frac{y\,\mathrm{d}y}{\sqrt{a^2-y^2}(x^2-y^2)}\,\mathcal{L}\left(\frac{yr}{x^2}\right)\psi(y,\phi)$$

$$= \mathcal{L}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2 - r^2}}\,\mathcal{L}\!\left(\frac{1}{\rho}\right) v(\rho,\phi). \tag{3.8.8}$$

One can interchange the order of integration in the second term of (8) and perform the integration with respect to x. The result is

$$-\psi(r,\phi) + \frac{1}{\pi^3}\int_0^{2\pi}\int_b^a \frac{K(y,r,\phi-\phi_0) - K(r,y,\phi-\phi_0)}{r^2 - y^2}\,\psi(y,\phi_0)\,\mathrm{d}y\,\mathrm{d}\phi_0$$

$$= \mathcal{L}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2 - r^2}}\,\mathcal{L}\!\left(\frac{1}{\rho}\right) v(\rho,\phi). \tag{3.8.9}$$

The kernel of (9) can be expressed in terms of elementary functions as follows:

$$K(y,r,\phi-\phi_0) = ry\left(\frac{a^2 - r^2}{a^2 - y^2}\right)^{1/2}\left\{\lambda\!\left(\frac{r}{y},\phi-\phi_0\right)\frac{1}{r}\ln\!\left(\frac{a+r}{a-r}\right)\right.$$

$$\left. -\,2\Re\left[\frac{1}{\xi\left(1 - \frac{r}{y}\mathrm{e}^{-i(\phi-\phi_0)}\right)}\ln\frac{a+\xi}{a-\xi}\right]\right\}, \tag{3.8.10}$$

where

$$\xi = \sqrt{yr\mathrm{e}^{i(\phi-\phi_0)}}. \tag{3.8.11}$$

Here $\Re$ denotes the real part of the expression to follow. Thus, the general problem of annular punch has been reduced to a Fredholm integral equation (9) with an elementary kernel which can be solved numerically. It is noteworthy that the governing equation for each specific harmonic will also have an elementary kernel. For example, the equation corresponding to the zero harmonic is

$$-\psi_0(r) + \frac{2}{\pi^2}\int_b^a \frac{K_0(y,r) - K_0(r,y)}{r^2 - y^2}\,\psi_0(y)\,\mathrm{d}y = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{v_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{\rho^2 - r^2}}, \tag{3.8.12}$$

with

$$K_0(y,r) = y\left(\frac{a^2-r^2}{a^2-y^2}\right)^{1/2} \ln\frac{a+r}{a-r}. \tag{3.8.13}$$

There have been so many variations of the governing integral equation published for the case of axial symmetry, that there is no doubt that equation (12) coincides with some known result, though we have difficulty to pinpoint exactly which one. The governing integral equation for the first harmonic will take the form

$$-\psi_1(r) + \frac{2}{\pi^2}\int_b^a \frac{K_1(y,r)-K_1(r,y)}{r^2-y^2}\,\psi_1(y)\,\mathrm{d}y = \frac{1}{2\pi}\,r\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{v_1(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}, \tag{3.8.14}$$

with

$$K_1(y,r) = \left(\frac{a^2-r^2}{a^2-y^2}\right)^{1/2}\left[\frac{y^2}{r}\ln\frac{a+r}{a-r} - 2a\right]. \tag{3.8.15}$$

There is no need to compute the charge distribution σ if one is interested in the integral characteristics only. Indeed, both the total charge Q and the moment M can be expressed through the new unknown function ψ as follows:

$$Q = \frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{\psi(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a^2-\rho^2}} = 4\int_b^a \frac{\psi_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}}, \tag{3.8.16}$$

$$M = -\frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{\psi(\rho,\phi)\,\rho^2\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a^2-\rho^2}} = -2\int_b^a \frac{\psi_1(\rho)\,\rho^2\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}}. \tag{3.8.17}$$

We note also that the kernels in (12) and (14) are finite at the point $y=r$. The following limits can be computed

$$\lim_{y\to r}\frac{K_0(y,r)-K_0(r,y)}{r^2-y^2} = \frac{1}{a^2-r^2}\left[a - \frac{r^2+a^2}{2r}\ln\frac{a+r}{a-r}\right],$$

$$\lim_{y\to r}\frac{K_1(y,r)-K_1(r,y)}{r^2-y^2} = \frac{1}{a^2-r^2}\left[3a - \frac{3a^2-r^2}{2r}\ln\frac{a+r}{a-r}\right]. \tag{3.8.18}$$

Equations (9), (10), (16), and (17) are the main new results of this section. The integral equations can be solved by any regular numerical method.

A similar approach can be extended to spherical coordinates. We consider a Dirichlet problem for a spherical annulus $\beta\le\theta\le\alpha$, $0\le\phi<2\pi$. The annulus is a part of a sphere of radius a. Let an arbitrary potential $v(\theta,\phi)$ be prescribed at the annulus, with no charges elsewhere. We need to find the charge density distribution σ at the spherical annulus. The governing integral equation may be written as

$$\int\int_S \frac{\sigma}{R}\,dS \equiv \frac{a}{2\cos(\theta/2)}$$

$$\times \int_\beta^\alpha \int_0^{2\pi} \frac{\sigma(\theta_0,\phi_0)\sin\theta_0\,d\theta_0\,d\phi_0}{\cos(\theta_0/2)\sqrt{\tan^2(\theta/2)+\tan^2(\theta_0/2)-2\tan(\theta/2)\tan(\theta_0/2)\cos(\phi-\phi_0)}} = v(\theta,\phi). \tag{3.8.19}$$

By using the integral representation for the reciprocal of the distance (1.5.3) for $r=a$, equation (19) will take the form

$$\frac{a}{\pi}\Bigg[\int_\beta^\theta \frac{d\tau}{\sqrt{\cos\tau-\cos\theta}}\int_\tau^\alpha \frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\tau-\cos\theta_0}}\,\mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi_0)$$

$$+\int_0^\beta \frac{d\tau}{\sqrt{\cos\tau-\cos\theta}}\int_\beta^\alpha \frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\tau-\cos\theta_0}}\,\mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi_0)\Bigg] = v(\theta,\phi). \tag{3.8.20}$$

Application of the operator

$$\mathcal{L}[\cot(\theta_1/2)]\frac{d}{d\theta_1}\int_\beta^{\theta_1} \frac{\sin\theta\,d\theta}{\sqrt{\cos\theta-\cos\theta_1}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right)$$

to both sides of (20) leads to

$$\frac{a}{\pi}\Bigg\{\pi\int_{\theta_1}^\alpha \frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_1-\cos\theta_0}}\,\mathcal{L}\left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi)$$

$$+\int_0^\beta \frac{\sin\theta_1\sqrt{\cos\tau-\cos\beta}\,d\tau}{(\cos\tau-\cos\theta_1)\sqrt{\cos\beta-\cos\theta_1}}$$

$$\times \int_{\beta}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\tau - \cos\theta_0}} \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta_1/2)\tan(\theta_0/2)} \right) \sigma(\theta_0, \phi) \Bigg\}$$

$$= \mathcal{L}[\cot(\theta_1/2)] \frac{d}{d\theta_1} \int_{\beta}^{\theta_1} \frac{\sin\theta\, d\theta}{\sqrt{\cos\theta - \cos\theta_1}} \mathcal{L}\left(\tan\frac{\theta}{2} \right) v(\theta, \phi). \tag{3.8.21}$$

Introduction of a new unknown

$$\chi(\theta_1, \phi) = \int_{\theta_1}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_1 - \cos\theta_0}} \mathcal{L}\left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)} \right) \sigma(\theta_0, \phi)$$

will lead to the new type of integral equations which would be the spherical equivalent to (9). This exercise is left to the reader.

CHAPTER 4

APPLICATIONS IN DIFFUSION AND ACOUSTICS

4.1. Diffusion through perforated membranes

The diffusion process through a thin membrane, perforated by several holes of arbitrary shape, is considered. A general theorem is established which relates the total flux through each hole with the concentration distribution of some chemical species, prescribed in the hole, by a system of linear algebraic equations. The theorem is applied to the case of arbitrarily located elliptical holes. Several specific examples are considered.

Advances in bioengineering have generated wide interest in the diffusion mechanism of biological membranes. An exact solution to the problem of steady-state diffusion through an isolated circular hole had been found many years ago (Rayleigh, 1948), along with a very good approximate solution for the case of a thick membrane. There seems to be no exact solutions for the case of several holes. Some quantitative considerations on this subject were made by Lord Rayleigh (1948) who indicated that the interaction between the holes should result in *decreasing* of the flux through each hole while the total flux should increase. The limiting case, when the number of holes tends to infinity, results in the total flux tending to infinity while the flux through each hole tends to zero. In spite of these well established facts, several papers have been published (S. Prager and H.L. Frish, J. Chem. Phys. **62**, 89 (1975); G.M. Malone, Suk Youn Suh, and S. Prager, in *Lecture Notes in Mathematics* **1035**, (1983), pp. 370-390.) in which the result is quite opposite: the holes' interaction *increases* the flux through each hole! Since the authors of these papers never explain the reason for such a discrepancy, one should think that those papers are just incorrect.

Here, a general theorem is established which is valid for arbitrary holes, and relates the flux through each hole, with the chemical species concentration prescribed inside the hole, to a system of linear algebraic equations. Since the coefficients of these equations depend on the concentration outside a hole of arbitrary shape, the theorem can be used effectively for elliptical holes only, as there seems to be no publication giving the concentration outside a nonelliptical hole. The exact values of the coefficients are not known but, since their variation is quite limited, the established theorem allows one to obtain the upper and the lower bounds for the parameters sought. There is no need to know the flux intensity distribution in the hole which presents a significant advantage for the user. Several examples are considered to show that these bounds may be so close that they provide, in fact, a reasonably accurate solution to the problem.

Theory. Consider a thin membrane conceived as the plane $z=0$, which separates two half-spaces, namely, $z>0$ and $z<0$. Assume that at $z\to\infty$ the concentration of some chemical species is $w^{+}=const.$, with the corresponding concentration $w^{-}=const.$ at $z\to-\infty$. The membrane is perforated by N holes of arbitrary shape and location. The analysis is limited to the steady-state diffusion through these holes.

For the sake of mathematical convenience, we can consider an equivalent problem with the limiting values $w=0$ for $z\to\infty$ and $w=w^{-}-w^{+}$ for $z\to-\infty$. This will enable us to use the Newton potential representation for the solution. The mathematical formulation of the problem reads: find a function W harmonic in the upper half-space ($\Delta W=0$) subject to the boundary conditions

$$W=w_n(M) \quad \text{for} \quad M\in S_n,$$

$$\sigma_n(M)=0 \quad \text{for} \quad M\notin S_n, \quad n=1,2,...,N. \tag{4.1.1}$$

where S_n denotes the area of the nth hole, and σ stands for the concentration gradient normal to the plane $z = 0$

$$\sigma=-\frac{\partial W}{\partial z}.$$

In our case $w_n=(w^{-}-w^{+})/2=\text{const.}$ but we are deliberately considering a more general case in order to show the applicability of the method. Using the Newton potential representation, we can write

$$W(Q)=D\sum_{n=1}^{N}\iint\limits_{S_n}\frac{\sigma_n(T_n)}{R(T_n,Q)}\,\mathrm{d}S_n. \tag{4.1.2}$$

where D is the mass diffusivity coefficient, and $R(T_n,Q)$ denotes the distance between a point $T_n\in S_n$ and a field point Q. Substitution of the boundary conditions (1) in (2) leads to a set of N integral equations. The exact solutions of these equations are not known at the present time even for the case of several circles. Here, we are to show that we do not really need to know these solutions. We can single out, without loss of generality, the first hole, and consider the related integral equation

$$w_1(Q_1) = D \iint_{S_1} \frac{\sigma_1(T_1)}{R(T_1,Q_1)}]\, \mathrm{d}S_1 + D \sum_{n=2}^{N} \iint_{S_n} \frac{\sigma_n(T_n)}{R(T_n,Q_1)}\, \mathrm{d}S_n. \tag{4.1.3}$$

Suppose that the function σ_0 is known, satisfying the following integral equation inside S_1

$$\iint_{S_1} \frac{\sigma_0(Q_1)}{R(T_1,Q_1)}\, \mathrm{d}S_1 = 1. \tag{4.1.4}$$

Multiplication of both sides of (3) by $\sigma_0(Q_1)$ and integration over the area S_1 yields

$$\iint_{S_1} \sigma_0(Q_1)\, w_1(Q_1)\, \mathrm{d}S_1 = D \iint_{S_1} \sigma_0(Q_1)\, \mathrm{d}S_1 \iint_{S_1} \frac{\sigma_1(T_1)}{R(T_1,Q_1)}]\, \mathrm{d}S_1$$

$$+ D \sum_{n=2}^{N} \iint_{S_1} \sigma_0(Q_1)\, \mathrm{d}S_1 \iint_{S_n} \frac{\sigma_n(T_n)}{R(T_n,Q_1)}\, \mathrm{d}S_n. \tag{4.1.5}$$

By changing the order of integration in (5) and taking into consideration that σ_0 satisfies (4), the following result can be obtained

$$\iint_{S_1} \sigma_0(Q_1)\, w_1(Q_1)\, \mathrm{d}S_1 = D\left[P_1 + \sum_{n=2}^{N} \iint_{S_n} w_{1n}(T_n)\, \sigma_n(T_n)\, \mathrm{d}S_n \right]. \tag{4.1.6}$$

where P_1 is the total flux through the first hole

$$P_1 = \iint_{S_1} \sigma_1(Q_1)\, \mathrm{d}S_1\,,$$

and

$$w_{1n}(T_n) = \iint_{S_1} \frac{\sigma_0(Q_1)}{R(T_n,Q_1)}\, \mathrm{d}S_1. \tag{4.1.7}$$

which is proportional to the concentration in the domain S_n due to a unit concentration of the chemical species in S_1. By evoking the mean value theorem which is valid when σ_n does not change sign, we come to the linear algebraic equation

$$\iint_{S_1} \sigma_0(Q_1)\, w_1(Q_1)\, \mathrm{d}S_1 = D\left[P_1 + \sum_{n=2}^{N} w_{1n}(C_n)\, P_n \right]. \tag{4.1.8}$$

The exact location of the point C_n is not known but the fact that $C_n \in S_n$ allows only limited variation within S_n, and in many cases provides sufficiently close upper and lower bounds for the parameters sought. By using the same logic, $N-1$ additional linear algebraic equations can be derived for the remaining holes. This set of equations provides the necessary relationships between the individual fluxes through the holes and the concentration gradients prescribed in the holes.

The set of linear algebraic equations of the type (8), is the main result of this section. In order to use (8), one needs to know the concentration distribution outside every hole in the system due to a unit concentration prescribed inside it which, at the moment, is available for the elliptical holes only.

Application to elliptical holes. Consider the interaction of a set of N elliptical holes arbitrarily located in an infinite thin membrane. Let a_n and b_n be the major and the minor semiaxes of the nth ellipse; X_n and Y_n be its center, and θ_n be the angle between the axis Ox and the major semiaxis a_n; P_n denotes the flux passing through the nth hole.

Here are some results implicitly given by Lur'e (1964). The function σ_0 has the form

$$\sigma_0 = \frac{1}{2\pi b_1 K(k_1)}\left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2}, \tag{4.1.9}$$

where $K(k_1)$ stands for the complete elliptic integral of the first kind, and k_1 is the eccentricity of the first ellipse

$$k_1 = \sqrt{1-(b_1/a_1)^2}. \tag{4.1.10}$$

Introduce the notation

$$R = \sqrt{(x-x_0)^2 + (y-y_0)^2},$$

$$Z_0 = \left[\, 1 - \frac{x_0^2}{a_1^2} - \frac{y_0^2}{b_1^2} \right]^{1/2}.$$

The following integrals are valid

$$\iint_{S_1} \frac{\mathrm{d}x_0 \mathrm{d}y_0}{RZ_0} = \begin{cases} 2\pi b_1 K(k_1), & \text{for } (x,y) \in S_1; \\ 2\pi b_1 F(\phi_1, k_1), & \text{for } (x,y) \notin S_1; \end{cases} \tag{4.1.11}$$

where

$$\phi_1 = \sin^{-1}\left[\frac{1}{\rho_1}\right], \tag{4.1.12}$$

$$\rho_1 = [L + (L^2 - k_1^2 x^2/a_1^2)^{1/2}]^{1/2}, \qquad L = \frac{1}{2}[k_1^2 + (x^2+y^2)/a_1^2]. \tag{4.1.13}$$

The remaining integral is elementary

$$\iint_{S_1} \left[\, 1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2} \mathrm{d}x\, \mathrm{d}y = 2\pi a_1 b_1. \tag{4.1.14}$$

The boundary conditions (1) in this case will take the form

$$w_n = (w^- - w^+)/2 = \delta. \tag{4.1.15}$$

Substitution of (15) into (8) yields

$$\frac{a_1}{K(k_1)}\delta = D\left[P_1 + \sum_{n=2}^{N} \frac{F(\phi_{1n}, k_1)}{K(k_1)} P_n \right], \tag{4.1.16}$$

where $K(k_1)$ and $F(\phi_{1n}, k_1)$ stand for the complete and incomplete elliptic integrals of the first kind ; ρ_{1n} and ϕ_{1n} are defined according to (12) and (13); formulae (9), (11), and (14) were used in the derivation of (16); Equation (16) represents the first of the set of N linear algebraic equations, which allows one to obtain the exact upper and lower bounds for the fluxes P_n, $n = 1,2, \ldots, N$ without having solved the system of integral equations (3). It is also important to note that each equation in the set is valid in the local system of the coordinates related to the center of the ellipse.

Two equal elliptical holes. Consider the case $N = 2$, $a_1 = a_2 = \mathrm{a}$, $b_1 = b_2 = \mathrm{b}$, $X_1 = Y_1 = 0$, $X_2 = \mathrm{h}$, $Y_2 = 0$, $\theta_1 = \theta_2 = 0$. Since $P_1 = P_2 = P$ then, due to the symmetry of the system, the set of equations, equivalent to (16), reduces to just one equation, namely

$$\frac{a}{K(k)}\delta = D\left[P + \frac{F(\phi,k)}{K(k)}P\right], \tag{4.1.17}$$

with an immediate result

$$P = \frac{P_0}{1 + \dfrac{F(\phi,k)}{K(k)}}. \tag{4.1.18}$$

where $P_0 = \delta a / DK(k)]$ indicates the flux through a solitary hole. Equation (18) shows that the interaction between the holes decreases the flux through each hole. The upper and the lower bounds for P can be obtained from (18) by taking

$$\phi = \sin^{-1}\left[\frac{a}{h-a}\right], \quad \text{and} \quad \phi = \sin^{-1}\left[\frac{a}{h+a}\right]. \tag{4.1.19}$$

respectively. We shall also consider the central estimation for P defined by

$$\phi = \sin^{-1}\left[\frac{a}{h}\right]. \tag{4.1.20}$$

Fig. 4.1.1 plots the ratio P/P_0 versus h/a for $a = 2$, $b = 1$. The solid line gives the upper bound, the broken line gives the lower bound, and the small circles plot the central estimation. Numerical computations show that the maximal error of the central estimation is less than 9% for $h/a > 3.5$, it is less than 5% for $h/a > 5$, it is less than 2% for $h/a > 8$, and it is less than 1% for $h/a > 12$. Since there is no accurate solution available for this case, it is difficult to say how great the real error of the central estimation is, but there is a reason to believe that it is much less than the one indicated above. The reason for such a belief comes from a comparison of the central estimation for two equal circular holes with the numerical solution by Kobayashi (1939) If one takes the Kobayashi's solution as exact then the maximum error of the central estimation does not exceed 0.4% in the whole range of $2 \le h/a < \infty$. Even if one assumes the accuracy in the case of two elliptic holes being ten times worse than the accuracy of the central estimation for two circular holes, this still would give the maximum error

Fig. 4.1.1. The ratio P/P_0 for two equal elliptical holes

at 4% which is not bad. Having this in mind, we shall evaluate the central estimation only in the examples to follow.

Four equal elliptical holes. Consider a rectangle with sides l and h. Locate the centers of four equal elliptical holes at its apices, with their axes being along the sides of the rectangle (Fig. 4.1.2). The holes are numbered in the clockwise direction. Due to the symmetry of the system, it is sufficient to consider just one equation of the set (16). The result is

$$\frac{a}{K(k)}\delta = DP\left[\,1+\frac{F(\phi_{12},k)}{K(k)}+\frac{F(\phi_{13},k)}{K(k)}+\frac{F(\phi_{14},k)}{K(k)}\,\right], \tag{4.1.21}$$

where

$$\phi_{1n} = \sin^{-1}\frac{1}{\rho_{1n}}, \qquad \text{for} \quad n = 2,3,4\,; \tag{4.1.22}$$

$$\rho_{12} = [1+(h^2-b^2)/a^2]^{1/2}, \qquad \rho_{14} = \frac{l}{a},$$

$$\rho_{13} = [L+\sqrt{L^2-k^2l^2/a^2}]^{1/2}, \quad L = \frac{1}{2}[k^2+(l^2+h^2)/a^2]. \tag{4.1.23}$$

We can note certain relation between our results and those of Grinberg (1948) who established some relevant theorems in electrostatics. It is also of interest to indicate certain limiting cases. In the case of a circular hole the eccentricity $k_1 \to 0$, and

Fig. 4.1.2. Four equal elliptical holes

$$\frac{F(\phi_{1n},k_1)}{K(k_1)} \to \frac{2}{\pi}\sin^{-1}\left(\frac{a_1}{r_{1n}}\right). \qquad (4.1.24)$$

and equation (16) will take the form

$$\frac{2}{\pi}a_1\delta_1 = D\left[P_1 + \frac{2}{\pi}\sum_{n=2}^{N} P_n \sin^{-1}\frac{a_1}{r_{1n}}\right], \qquad (4.1.25)$$

where a_1 is the radius of the hole number one and r_{1n} is the distance from the center of the first hole to the nth hole. Formula (25) is in agreement with the result by Fabrikant (1985)

4.2. Interaction between circular pores

By using a special integral representation for the Green's function, a system of Fredholm integral equations is derived with respect to the flux density. Equations are non-singular and allow an accurate numerical solution. The total flux can be found from a system of linear algebraic equations.

We consider a thin membrane $z=0$, perforated by n circular holes. The center of i-th hole is located at the point (x_i, y_i), and its radius is denoted a_i. We can formulate the problem of diffusion as that of finding a harmonic function W subject to the boundary conditions

$$W(\rho,\phi,0) = w_i(\rho,\phi), \quad \text{for} \quad i=1,2,...,n, \ \text{and} \quad (\rho,\phi)\in S_i;$$

$$\frac{\partial W}{\partial z} \equiv -\sigma = 0 \quad \text{for} \ z=0 \ \text{and} \quad (\rho,\phi) \notin S_i.$$

Here S_i is the area of i-th pore, σ is the flux density. The mass diffusivity was assumed to be unity. We can single out, without loss of generality, the pore number 1, and locate the origin of polar coordinates at its center. The governing integral equation for the first pore can be written as

$$w_1(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{a_1} \frac{\sigma_1(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}$$

$$+\frac{1}{2\pi}\sum_{k=2}^{n}\iint_{S_k} \frac{\sigma_k(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}. \tag{4.2.1}$$

Substituting the integral representation (1.2.22) in (1), we obtain

$$w_1(\rho,\phi) = \frac{2}{\pi}\int_0^{\rho} \frac{dx}{(\rho^2-x^2)^{1/2}}\int_x^{a_1} \frac{\rho_0\,d\rho_0}{(\rho_0^2-x^2)^{1/2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma_1(\rho_0,\phi)$$

$$+\frac{1}{\pi^2}\sum_{k=2}^{n}\int_0^{\rho} \frac{dx}{(\rho^2-x^2)^{1/2}}\iint_{S_k} \frac{\rho_0\,d\rho_0}{(\rho_0^2-x^2)^{1/2}}\,\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\sigma_k(\rho_0,\phi_0)\,d\phi_0. \tag{4.2.2}$$

We apply the integral operator

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_0^{r} \frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)$$

to both sides of (2). The result is

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w_1(\rho,\phi) = \int_r^{a_1} \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}\,\mathcal{L}\left(\frac{r}{\rho_0}\right)\sigma_1(\rho_0,\phi)$$

$$+\frac{1}{2\pi}\sum_{k=2}^{n}\int\int_{S_k} \frac{\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\sqrt{\rho_0^2-r^2}}\,\lambda\left(\frac{r}{\rho_0},\,\phi-\phi_0\right)\sigma_k(\rho_0,\phi_0). \tag{4.2.3}$$

The next operator to apply is

$$\mathcal{L}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1} \frac{r\,\mathrm{d}r}{\sqrt{r^2-t^2}}\,\mathcal{L}\left(\frac{1}{r}\right).$$

The final result is

$$-\frac{2}{\pi}\,\mathcal{L}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1} \frac{r\,\mathrm{d}r}{\sqrt{r^2-t^2}}\,\mathcal{L}\left(\frac{1}{r^2}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w_1(\rho,\phi) = \sigma_1(t,\phi)$$

$$+\frac{1}{\pi^2\sqrt{a_1^2-t^2}}\sum_{k=2}^{n}\int\int_{S_k} \frac{\sqrt{\rho_0^2-a_1^2}\,\sigma_k(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{t^2+\rho_0^2-2t\rho_0\cos(\phi-\phi_0)}. \tag{4.2.4}$$

Similar equations can be obtained for the remaining $n-1$ pores. The integral equations are non-singular, and can be solved by any regular numerical method. In the case when $w_1 = c_0$ is a constant, the governing integral equation will take the form

$$\sigma_1(\rho,\phi) = \frac{2c_0}{\pi\sqrt{a_1^2-\rho^2}} - \frac{1}{\pi^2\sqrt{a_1^2-\rho^2}}\sum_{k=2}^{n}\int\int_{S_k} \frac{\sqrt{\rho_0^2-a_1^2}\,\sigma_k(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\,\rho_0\cos(\phi-\phi_0)}. \tag{4.2.5}$$

If we need to obtain the total flux only, we do not have to solve integral equations (5). Indeed, we can multiply both sides of (5) by $\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$ and integrate over the first pore. The result is

$$P_1 = 4a_1c_0 - \frac{2}{\pi}\sum_{k=2}^{n}\int\int_{S_k} \sigma_k(\rho_0,\phi_0)\sin^{-1}\left(\frac{a_1}{\rho_0}\right)\rho_0\, d\rho_0\, d\phi_0. \tag{4.2.6}$$

Here P_1 is the flux through the first pore. Note that the relationship (6) is exact. We can now apply the mean value theorem which is valid when σ_k does not change sign, and get the set of linear algebraic equations with respect to the total fluxes

$$P_k + \frac{2}{\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} P_i \sin^{-1}\left(\frac{a_k}{b_{ki}}\right) = 4a_kc_0, \quad \text{for} \quad k = 1, 2, ..., n. \tag{4.2.7}$$

Here

$$r_{ki} - a_i \le b_{ki} \le r_{ki} + a_i, \quad r_{ki} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}. \tag{4.2.8}$$

Of course, the exact value of b_{ki} is not known, but the set of equations (7) can be used to obtain the upper and the lower bounds for the flux by simple variation of b_{ki} in the admissible range (8). In general $b_{ki} \neq b_{ik}$, except for the case $a_k = a_i$. Due to the reciprocal theorem, it can be shown that for $a_k < a_i$ the admissible range for b_{ki} becomes more narrow which might improve the flux estimation. In certain cases the flux estimation for unequal holes is sharper than that for the equal ones. Since the case of equal pores might be the least accurate, it is analyzed in the examples to follow.

Two circular pores. Let a_1 and a_2 be the radii of the holes, the distance between their centers being b. The system (7) in this case takes the form

$$P_1 + \frac{2}{\pi}\sin^{-1}\left(\frac{a_1}{b_{12}}\right) = 4\, a_1\, c_0,$$

$$\frac{2}{\pi}P_1 \sin^{-1}\left(\frac{a_2}{b_{21}}\right) + P_2 = 4\, a_2\, c_0.$$

The solution is

$$P_1 = 4c_0 \frac{a_1 - \frac{2}{\pi} a_2 \sin^{-1}\left(\frac{a_1}{b_{12}}\right)}{1 - \frac{4}{\pi^2} \sin^{-1}\left(\frac{a_1}{b_{12}}\right) \sin^{-1}\left(\frac{a_2}{b_{21}}\right)},$$

$$P_2 = 4c_0 \frac{a_2 - \frac{2}{\pi} a_1 \sin^{-1}\left(\frac{a_2}{b_{21}}\right)}{1 - \frac{4}{\pi^2} \sin^{-1}\left(\frac{a_1}{b_{12}}\right) \sin^{-1}\left(\frac{a_2}{b_{21}}\right)}.$$

When $a_1 = a_2 = a$, the solution simplifies to

$$P_1 = P_2 = \frac{4c_0 a}{1 + \frac{2}{\pi} \sin^{-1}\left(\frac{a}{b_{12}}\right)} = \frac{P_0}{1 + \frac{2}{\pi} \sin^{-1}\left(\frac{a}{b_{12}}\right)}. \tag{4.2.9}$$

Here P_0 is the flux through an isolated hole and the denominator in (9) shows the degree of the flux reduction due to the second hole. When the distance $b \to \infty$, the two holes do not interact. The smallest possible value is given by $b = 2a$, and the flux through each hole will be $0.75P_0$. Computations were performed in Fabrikant (1985) which showed a very good accuracy when b_{12} was taken equal to b. This is called the central estimation. The different estimations for the dimensionless flux $P^* = P/P_0$ is given in Table 4.2.1.

General configuration. We consider the case of N equal holes of radius a located at the apexes of a regular polygon. The hole of radius a_0 is located at the geometric center of the polygon. Let the distance between the center of any apex be h. From geometric consideration, the distance between the n-th and k-th apex is ely,

$$P_0 + \frac{2}{\pi} N P \sin^{-1}\left(\frac{a_0}{h}\right) = 4a_0 c_0,$$

$$\frac{2}{\pi} P_0 \sin^{-1}\left(\frac{a}{h}\right) + P\left[1 + \frac{2}{\pi} \sum_{n=1}^{N-1} \sin^{-1}\left(\frac{a}{2h \sin(\pi n/N)}\right)\right] = 4ac_0.$$

The solution is

Table 4.2.1. Comparison of our results with Kobayashi's (1939)

Distance between the centres	Upper bound for the flux F^*	Lower bound for the flux F^*	Central estimation of F^*	Kobayashi's result	Our numerical result
2.0	0.8221338	0.5000000	0.7500000	0.75272	0.75239
2.2	0.8317164	0.6145749	0.7689962	0.77014	0.76996
2.4	0.8402998	0.6637918	0.7851738	0.78545	0.78537
2.6	0.8480356	0.6993975	0.7991486	0.79898	0.79893
2.8	0.8550458	0.7272786	0.8113602	0.81096	0.81093
3.0	0.8614294	0.7499999	0.8221338	0.82162	0.82161
3.5	0.8751494	0.7924057	0.8442654	0.84370	0.84370
4.0	0.8863767	0.8221338	0.8614294	0.86093	0.86092
5.0	0.9036683	0.8614294	0.8863767	0.88602	0.88602
7.0	0.9261093	0.9036683	0.9163737	0.91619	0.91620
10.0	0.9452202	0.9338098	0.9400541	0.93999	0.93998
∞	1.0	1.0	1.0	1.0	1.0

$$P_0 = 4c_0 \frac{a_0\left[1+\frac{2}{\pi}\sum_{n=1}^{N-1}\sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right]-\frac{2}{\pi}aN\sin^{-1}\left(\frac{a_0}{h}\right)}{\left[1+\frac{2}{\pi}\sum_{n=1}^{N-1}\sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right]-\frac{4}{\pi^2}N\sin^{-1}\left(\frac{a}{h}\right)\sin^{-1}\left(\frac{a_0}{h}\right)},$$

$$P = 4c_0 \frac{a-\frac{2}{\pi}a_0\sin^{-1}\left(\frac{a}{h}\right)}{\left[1+\frac{2}{\pi}\sum_{n=1}^{N-1}\sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right]-\frac{4}{\pi^2}N\sin^{-1}\left(\frac{a}{h}\right)\sin^{-1}\left(\frac{a_0}{h}\right)}.$$

The same method can be used for membranes of finite thickness, and it is presented in the next section.

4.3. Pore length effect

In the previous sections the membrane was assumed to be infinitely thin. The diffusion of chemical species through a pore of finite length, including entrance and exit effects, has important applications to transport and filtration processes in biological membranes or membrane-like structures. We consider now a system comprising of a thick layer penetrated by discrete uniform pores, which are approximated as identical right cylinders of radius a and length $2l$ (Fig. 4.3.1). The layer is sandwiched between two stagnant fluids of infinite

Fig. 4.3.1. Geometry of the problem

extent. The solute concentration far from the pore in either fluid is held constant (say, at w^+ and w^- respectively). The problem is to find a steady-state concentration in the combined space.

The method of solution proposed by Kelman (1965) involves the use of oblate spheroidal coordinates in the half-space and the polar cylindrical coordinated inside the pore. In order to satisfy the matching conditions at the pore opening, one has to expand each term of an infinite sum representing the concentration, into yet another infinite sum in terms of the other system of coordinates. All this makes the method very cumbersome, and its accuracy doubtful.

An alternative approach is presented here. It uses the closed form representation for the concentration in region I (half-space) by utilizing the Green's function for a half-space. The Fourier-Bessel series representation is used in region II (pore). Some simple relations involving concentration and its normal derivatives can be established. Such relations will allow us to arrive at the general form for permeability P, expressions for the local flux and concentration profile. Due to symmetry, it is sufficient to focus attention on half the system, i.e. a pore of radius a and length $2l$ with an infinite region above the pore.

Let ρ, ϕ, and z denote cylindrical coordinates measured from the pore opening. The problem to solve now reads

$$\Delta w = 0 \quad \text{for} \quad -l < z < 0 \quad \text{and} \quad 0 < \rho < a; \quad \text{or} \quad z > 0 \quad \text{and} \quad \rho > 0,$$

subject to the boundary conditions

$$\frac{\partial w}{\partial \rho} = 0 \quad \text{for} \ \ \rho = 0, \ \ z > -l, \text{ or } \ \ \rho = a, \ \ -l < z < 0,$$

$$\frac{\partial w}{\partial z} = 0 \quad \text{for} \ \ z = 0, \ \ \rho > a,$$

$$w = \frac{w^+ + w^-}{2} \quad \text{for} \ \ z = -l, \ \ 0 \le \rho < a,$$

$$w \to w^+ \quad \text{for} \ \ \sqrt{\rho^2 + z^2} \to \infty, \ \ z > 0.$$

The solution in the half-space $z \ge 0$ can be presented as

$$w(\rho, z) = w^+ + \frac{1}{2\pi} \int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0)\, \rho_0 d\rho_0 d\phi_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2}}. \tag{4.3.1}$$

As before,

$$\sigma(\rho) = -\frac{\partial w}{\partial z} \quad \text{at} \quad z = 0.$$

The solution in the pore region may be presented as

$$w(\rho, z) = \frac{w^+ + w^-}{2} + A_0(l + z) + \sum_{n=1}^{\infty} A_n \sinh\left(x_n \frac{l+z}{a}\right) J_0\left(x_n \frac{\rho}{a}\right). \tag{4.3.2}$$

Here x_n are positive roots of equation $J_1(x) = 0$, and A_n are the as yet unknown constants. Notice that both (1) and (2) satisfy the Laplace equation and the boundary conditions in their respective domains of validity, i.e. the half space and the cylinder respectively. The unknown constants are to be determined from the continuity conditions

$$w(\rho, 0^+) = w(\rho, 0^-); \qquad \frac{\partial w}{\partial z}\Big|_{z=0^+} = \frac{\partial w}{\partial z}\Big|_{z=0^-}. \tag{4.3.3}$$

Differentiation of (2) yields

$$\sigma(\rho) = -\left[A_0 + \sum_{n=1}^{\infty} A_n \frac{x_n}{a} \cosh\left(x_n \frac{l}{a}\right) J_0\left(x_n \frac{\rho}{a}\right)\right]. \tag{4.3.4}$$

By using integral representation (1.2.22) in the case of axial symmetry, equation (1) can be rewritten as

$$w^*(\rho) = \frac{2}{\pi}\int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\sigma(\rho_0)\,\rho_0\,d\rho_0}{(\rho_0^2 - x^2)^{1/2}}. \tag{4.3.5}$$

Here

$$w^* = w(\rho, 0) - w^+. \tag{4.3.6}$$

Application of the operator

$$\frac{d}{dr}\int_0^r \frac{\rho\,d\rho}{\sqrt{r^2 - \rho^2}} \tag{4.3.7}$$

to both sides of (5) yields

$$\frac{d}{dr}\int_0^r \frac{w^*(\rho)\,\rho\,d\rho}{\sqrt{r^2 - \rho^2}} = \int_r^a \frac{\sigma(\rho_0)\,\rho_0\,d\rho_0}{\sqrt{\rho_0^2 - r^2}}. \tag{4.3.8}$$

Substitution (2), (4), and (6) in (8) results in

$$A_0 l - \frac{w^+ - w^-}{2} + \sum_{n=1}^{\infty} A_n \sinh\left(x_n \frac{l}{a}\right)\cos\left(x_n \frac{r}{a}\right) = -\left[A_0\,(a^2 - r^2)^{1/2}\right.$$

$$\left. + \sum_{n=1}^{\infty} A_n \frac{x_n}{a} \cosh\left(x_n \frac{l}{a}\right) \int_r^a \frac{J_0(x_n\rho_0/a)\,\rho_0\,d\rho_0}{\sqrt{\rho_0^2 - r^2}}\right]. \tag{4.3.9}$$

Here the following integral was used (Gradshtein and Ryzhik, 1965):

$$\int_0^r \frac{J_0(x_n\rho/a)\,\rho\,d\rho}{\sqrt{r^2 - \rho^2}} = \frac{a}{x_n}\sin\left(x_n \frac{r}{a}\right). \tag{4.3.10}$$

The remaining integrals are elementary. Integration of both sides of (9) with respect to r from 0 to a gives

$$A_0\left(l+\frac{\pi}{4}a\right)+\sum_{n=1}^{\infty} A_n \sinh\left(x_n \frac{l}{a}\right)\frac{\sin x_n}{x_n}=\frac{1}{2}(w^+-w^-). \tag{4.3.11}$$

Multiplication of both sides of (9) by $\cos(x_k r/a)$ and subsequent integration with respect to r from 0 to a results in

$$A_0\, l\,\frac{\sin x_k}{x_k}+\frac{1}{2}\sum_{\substack{n=1\\ n\neq k}}^{\infty} A_n \sinh\left(x_n \frac{l}{a}\right)\left[\frac{\sin(x_n-x_k)}{x_n-x_k}+\frac{\sin(x_n+x_k)}{x_n+x_k}\right]$$

$$+\frac{1}{2}A_k \sinh\left(x_k \frac{l}{a}\right)\left[1+\frac{\sin(2x_k)}{2x_k}+\coth\left(x_k\frac{l}{a}\right)\frac{\pi x_k}{2}J_0^2(x_k)\right]=\frac{1}{2}(w^+-w^-)\frac{\sin x_k}{x_k}. \tag{4.3.12}$$

Here are some details of the transformations performed.

$$\int_0^a \cos\left(x_k\frac{r}{a}\right)\mathrm{d}r\int_r^a \frac{J_0(x_n\rho_0/a)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}=\int_0^a J_0(x_n\rho_0/a)\,\rho_0\,\mathrm{d}\rho_0\int_0^{\rho_0}\frac{\cos\left(x_k\frac{r}{a}\right)\mathrm{d}r}{\sqrt{\rho_0^2-r^2}}$$

$$=\frac{\pi}{2}\int_0^a J_0(x_n\rho_0/a)\,J_0(x_k\rho_0/a)\,\rho_0\,\mathrm{d}\rho_0=\begin{cases}\dfrac{\pi}{4}[aJ_0(x_k]^2 & \text{for } n=k\\[2ex] 0 & \text{for } n\neq k\end{cases} \tag{4.3.13}$$

$$\int_0^a (a^2-r^2)^{1/2}\cos\left(x_k\frac{r}{a}\right)\mathrm{d}r=a^2J_1(x_k)=0. \tag{4.3.14}$$

Now the constants A_0, A_1, ..., may be found from the infinite system of algebraic equations (11) and (12). By introducing the notation

$$c_0=\frac{w^+-w^-}{2},\quad \lambda=\frac{l}{a},\quad B_0=A_0\,l,\quad B_n=A_n\sinh\left(x_n\frac{l}{a}\right),\quad \xi_n=\frac{\sin x_n}{x_n},\quad n=1,2,....,\tag{4.3.15}$$

the system (11)–(12) can be rewritten in a more compact form, namely,

$$B_0\left(1+\frac{\pi}{4\lambda}\right)+\sum_{n=1}^{\infty} B_n\,\xi_n=c_0, \tag{4.3.16}$$

$$B_0\xi_k + \sum_{n=1}^{\infty} B_n q_{nk} = c_0\xi_k, \quad \text{for} \quad k = 1, 2, ..., \tag{4.3.17}$$

Here

$$q_{nk} = \frac{1}{2}\left[\frac{\sin(x_n - x_k)}{x_n - x_k} + \frac{\sin(x_n + x_k)}{x_n + x_k}\right], \quad \text{for} \quad n \neq k;$$

and

$$q_{nk} = \frac{1}{2}\left[1 + \frac{\sin(2x_k)}{2x_k} + \coth(x_k\lambda)\frac{\pi x_k}{2}J_0^2(x_k)\right], \quad \text{for} \quad n = k. \tag{4.3.18}$$

Notice that the matrix of the system (16)–(17) is symmetric, and that only diagonal elements depend on the value of the ratio $(l/a) = \lambda$, and the diagonal elements are dominating. The off-diagonal elements depend only on the values of the roots x_k, and they are decreasing with the distance from diagonal. These features make the system (16)–(17) very well-behaved, and guarantee high accuracy for any truncated system. Actual computations were made with $N = 100$. The most interesting constant is A_0 since it is proportional to the flux. The ratio F^* of total flux F through a pore of length l to the flux F_0 through an infinitely thin membrane is given in Fig. 4.3.2.

Some simple approximate formulae can be obtained as follows. We consider a truncated system of just two equations

$$B_0\left(1 + \frac{\pi}{4\lambda}\right) + B_1\xi_1 = c_0,$$

$$B_0\xi_1 + B_1 q_{11} = c_0\xi_1. \tag{4.3.19}$$

The solution is

$$B_0 = A_0 l = c_0 \frac{q_{11} - \xi_1^2}{q_{11}\left(1 + \frac{\pi}{4\lambda}\right) - \xi_1^2}, \tag{4.3.20}$$

$$B_1 = A_1\sinh(\lambda x_1) = c_0 \frac{\frac{\pi}{4\lambda}\xi_1}{q_{11}\left(1 + \frac{\pi}{4\lambda}\right) - \xi_1^2}. \tag{4.3.21}$$

Since $x_1 = 3.8317$, and $J_0(x_1) = -0.4028$, their substitution in (20) gives

Fig. 4.3.2. Dimensionless flux through a pore of length l

$$A_0 = \frac{c_0}{\left(l + \frac{\pi}{4}a\right)\eta - \frac{0.0276\, l}{0.5365 + 0.4883\coth(\lambda x_1)}}. \tag{4.3.22}$$

Here

$$\eta = \frac{0.5641 + 0.4883\coth(\lambda x_1)}{0.5365 + 0.4883\coth(\lambda x_1)}. \tag{4.3.23}$$

A very simple analysis shows that the value of η is almost constant, being unity for $\lambda \to 0$ and $\eta = 1.0269$ for $\lambda \to \infty$. Hence, a simple formula may be suggested for A_0, namely,

$$A_0 = \frac{4c_0}{\pi a}\frac{1}{1 + \frac{4}{\pi}\lambda}. \tag{4.3.24}$$

Formula (24) is exact when $\lambda = 0$, and its overall performance is good. The maximum error is about 2%, and it is achieved near $\lambda = 0.5$. Formula of Kelman (1965) has its maximum error of 5% for $\lambda = 0$, though for $\lambda > 0.5$ its error is less than 1%. Kelman's formula may be recommended for large λ when high accuracy is necessary, otherwise (24) may be used in the whole range $0 \le \lambda < \infty$. A very accurate formula (the maximum error less than 0.15%) can be obtained by a curve-fitting technique, namely,

$$F^* \equiv \frac{F}{F_0} = \frac{1}{1 + \frac{4}{\pi}\lambda + \frac{1}{21.4479 + 0.2564 \coth(0.3439\lambda)}}.$$

Concentration profiles. Finding of σ from (4) is difficult due to a bad convergence. This can be illustrated by the limiting case $\lambda=0$. The exact solution in this case is known:

$$w^*(\rho,z) = -\frac{2}{\pi} c_0 \sin^{-1}\left(\frac{a}{l_2}\right), \quad l_2 = \frac{1}{2}\{\sqrt{(a+\rho)^2+z^2} + \sqrt{(a-\rho)^2+z^2}\},$$

$$\frac{\partial w}{\partial z}\Big|_{z=0} = \frac{2c_0}{\pi(a^2-\rho^2)^{1/2}}. \tag{4.3.25}$$

On the other hand, the exact solution of (16)–(17) in this case is

$$A_0 = 4\frac{c_0}{\pi a}, \quad A_k = \frac{4c_0 \sin x_k}{\pi x_k^2 J_0^2(x_k)}, \quad \text{for} \quad k = 1, 2, ..., \tag{4.3.26}$$

By using integrals (10) and (13), we can show the validity of an expansion

$$A_0 + \sum_{n=1}^{\infty} A_n \frac{x_n}{a} J_0\left(x_n \frac{\rho}{a}\right) = \frac{4c_0}{\pi a}\left\{1 + \sum_{n=1}^{\infty} \frac{\sin x_n}{x_n J_0^2(x_n)} J_0\left(x_n \frac{\rho}{a}\right)\right\} = \frac{2c_0}{\pi(a^2-\rho^2)^{1/2}}. \tag{4.3.27}$$

Substitution of numerical values of x_n into (27) shows that the term $\sin x_n/[x_n J_0^2(x_n)]$ is approximately equal $(-1)^k$, and slightly increasing with k. This makes the expansion (27) practically non-convergent. This is true for any λ, so that expressions (4) and (1) become unfit for use.

An alternative approach is based on the solution of (5) which has the form

$$\sigma(\rho) = -\frac{2}{\pi\rho}\frac{d}{d\rho}\int_\rho^a \frac{t\,dt}{(t^2-\rho^2)^{1/2}}\frac{d}{dt}\int_0^t \frac{w^*(\rho_0)\,\rho_0\,d\rho_0}{\sqrt{t^2-\rho_0^2}}. \tag{4.3.28}$$

The last expression can be rewritten as

$$\sigma(\rho) = \frac{2}{\pi}\left[\frac{f(a)}{(a^2-\rho^2)^{1/2}} - \int_a^\rho \frac{df(t)}{\sqrt{t^2-\rho^2}}\right], \tag{4.3.29}$$

with

$$f(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{w^*(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{t^2-\rho_0^2}}=A_0\,l-c_0+\sum_{n=1}^{\infty} A_n\sinh(x_n\lambda)\cos\!\left(x_n\frac{t}{a}\right). \tag{4.3.30}$$

Substitution of (30) in (29) yields

$$\sigma(\rho)=\frac{2}{\pi}\left[\frac{A_0\,l-c_0+\sum_{n=1}^{\infty} A_n\sinh(x_n\lambda)\cos(x_n)}{(a^2-\rho^2)^{1/2}}+\sum_{n=1}^{\infty} A_n\sinh(x_n\lambda)\frac{x_n}{a}\int_\rho^a \frac{\sin(x_n t/a)\,\mathrm{d}t}{\sqrt{t^2-\rho^2}}\right]. \tag{4.3.31}$$

The convergence of (31) is better than that of (5), especially close to $\rho=a$, where the first term becomes dominating. The results of computation of the dimensionless local flux $\sigma^*=|\sigma|\pi a/(2c_0)$ is given in Fig. 4.3.3.

Fig. 4.3.3. Local flux at the pore entrance

The value of w^* in the half-space $z \geq 0$ may be expressed as

$$w^*(\rho, z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0)\, \rho_0\, d\rho_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2}}. \tag{4.3.32}$$

Now we make use of the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^a \frac{\rho_0\, d\rho_0}{(a^2 - \rho_0^2)^{1/2} \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2}} = \sin^{-1}\left(\frac{a}{l_2}\right). \tag{4.3.33}$$

Substitution of (29) in (32) yields, after integration according to (33),

$$w^*(\rho, z) = \frac{2}{\pi} \left\{ f(a) \sin^{-1}\left(\frac{a}{l_2}\right) - \int_0^a \sin^{-1}\left(\frac{t}{l_2(t)}\right) \mathrm{d}f(t) \right\}. \tag{4.3.34}$$

Integration by parts in (34) gives the final result

$$w^*(\rho, z) = \frac{1}{\pi} \int_0^a \frac{\sqrt{l_2^2(t) - t^2}}{l_2^2(t) - l_1^2(t)} f(t)\, \mathrm{d}t. \tag{4.3.35}$$

The terms l_1 and l_2 were defined on many occasions, see, for example, (3.6.4). Now the complete numerical procedure may be outlined as follows. Solution of the truncated system (16)–(17) gives the values of A_k, $k = 0, \ , \ , \ldots, N-1$. Here N denotes the order of truncation. The next step is evaluation of the function $f(t)$ according to (30). Its substitution in (35) gives the value of $w^*(\rho, z)$ in the half-space, while formula (2) gives the relevant values in the pore. It is recommended that the derivative $\frac{\partial w^*}{\partial z}$ be evaluated by numerical differentiation procedure. When ρ is close to a and $z = 0$, formula (31) is appropriate to use. Notice also that for $z = 0$, formula (35) changes to

$$w^*(\rho, 0) = \frac{2}{\pi} \int_0^{\min(\rho, a)} \frac{f(t)\, \mathrm{d}t}{\sqrt{\rho^2 - t^2}}. \tag{4.3.36}$$

Because of singularity when $\rho \leq a$, formula (36) is not good for numerical integration, and should be transformed to

$$w^*(\rho, 0) = \frac{2}{\pi}\int_0^{\pi/2} f(\rho \sin\phi)\, d\phi.$$

Fig. 4.3.4. Local concentration profile at the pore entrance

Fig. 4.3.5. Isoconcentration profiles: (a) $\lambda = 0$, (b) $\lambda = 0.5$, (c) $\lambda = 1.$, (d) $\lambda = 5.$

We note also that $w^*(0,0) = f(0)$. The results of computations are given in Fig. 4.3.4 and Fig. 4.3.5.

4.4. Sound transmission through an aperture in a rigid screen

Van Bladel has reduced the problem of low-frequency scattering through an aperture in a rigid screen to a sequence of static integral equations. Analytical solutions are known at the moment for a circular and an elliptic aperture only. A new analytical method is proposed here which is valid for the nonelliptical apertures. Specific approximate formulae are derived for evaluating the average value of the quadratic term in the low-frequency expansion for an aperture of general shape. Specific examples are considered. All the formulae are checked against the solutions known in the literature, and a good accuracy is confirmed.

The diffraction of a plane wave by an aperture in a rigid screen is an important acoustical problem. Though significant efforts were spent on the investigation of circular and elliptical apertures (see, for example, Van Bladel, 1967), very little is known about the apertures of general shape, except for some numerical solutions (De Smedt, 1981). Here we reproduce some essential results from (Van Bladel, 1967), which are necessary for better understanding of the new approach presented in this section.

Consider a flat rigid screen with a general aperture S. Let the incident field be a plane wave $P^i = \mathrm{e}^{-jkR}$, where $R = \mathbf{u}_i \cdot \mathbf{r}$, $\mathbf{u}_i$ is the incidence vector and $\mathbf{r}$ is the field point vector, the wave number $k = 2\pi/\lambda$, and λ is the wavelength. The governing integral equation takes the form

$$-\frac{1}{2\pi}\int\!\!\int_S \frac{\partial P}{\partial z_M}\frac{e^{-jkR(M,N)}}{R(M,N)}\,\mathrm{d}S_{\mathrm{M}} = P^i(N)$$

where P is the acoustic pressure in the aperture. In the low-frequency case, the characteristic length of the aperture is much smaller than the wavelength, and the following expansions become valid:

$$P(\mathbf{r}) = P_0(\mathbf{r}) + jkP_1(\mathbf{r}) + \frac{1}{2}(jk)^2 P_2(\mathbf{r}) + \ldots\ ,$$

$$-\frac{\partial P}{\partial z} = \alpha + jk\beta + \frac{1}{2}(jk)^2\gamma + \ldots.$$

Van Bladel (1967) has proven that the diffraction problem can be reduced to the solution of a sequence of integral equations of the following type:

$$w(N) = \iint_S \frac{\sigma(M)}{R(M,N)}\,\mathrm{d}S\,, \tag{4.4.1}$$

where S is a two-dimensional domain, $R(M,N)$ stands for the distance between the points M and N, w is a known function, and σ is the unknown function. If we denote σ_0, σ_x, σ_y, σ_{xx}, *etc.*, as solutions of (1), corresponding, respectively, to the function w taking on values $2\pi\sqrt{A}$, $2\pi x$, $2\pi x^2/\sqrt{A}$, *etc.*, where A is the area of the aperture, then the various parameters can be defined quite simply through these solutions. For example,

$$\alpha = \sigma_0/\sqrt{A},$$

$$\beta = -\mathbf{u}_i\cdot(\sigma_x\mathbf{u}_x + \sigma_y\mathbf{u}_y) + \frac{\sigma_0}{2\pi A}\iint_S \sigma_0\,\mathrm{d}S,\quad \ldots$$

The problem of sound penetration through an aperture was solved numerically for several specific shapes (De Smedt 1979). We shall use his results for the verification of the accuracy of the new method. The zeroth-order term in the low-frequency expansion was found analytically for an arbitrary aperture in (Fabrikant 1986c). The apparatus used there is essentially the same as that in section 3.3, so it is not repeated here. The first (linear) term can be found from section 3.4 where the mathematically equivalent problem of magnetic polarizability of small apertures of arbitrary shape was considered. Here, a similar method is used for the analysis of the quadratic term in the low-frequency expansion. The relevant theory is given further, with applications to specific aperture shapes (polygon, rectangle, rhombus, cross) to follow. The possibility of using the variational approach is discussed in the last part.

Theory. We outline the idea of the analytical treatment of such problems which allows the derivation of simple yet accurate formulae for various aperture shapes. The approach is based on the integral representation for the reciprocal distance between two points established in Chapter 1.

$$\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\mathrm{d}x}{(\rho^2-x^2)^{1/2}\,(\rho_0^2-x^2)^{1/2}}, \tag{4.4.2}$$

where

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}. \tag{4.4.3}$$

Substitution of (2) into (1) gives, after changing the order of integration

$$w(\rho,\phi)=\frac{2}{\pi}\int_0^{\rho}\frac{dx}{(\rho^2-x^2)^{1/2}}\int_0^{2\pi}d\phi_0\int_x^{a(\phi_0)}\frac{\lambda(\frac{x^2}{\rho\rho_0},\phi-\phi_0)}{(\rho_0{}^2-x^2)^{1/2}}\,\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0. \qquad (4.4.4)$$

Despite the fact that (4) is valid only inside a circle inscribed into the aperture, it will be shown further that expression (4) allows us to obtain approximate solutions of high accuracy for various aperture shapes.

Consider an aperture of general shape in a rigid screen. Let the boundary of the aperture S be given in the polar coordinates as

$$\rho = a(\phi),$$

where the function $a(\phi)$ is bounded and single-valued. Let the known function w take the form

$$w = g_x y^2 + g_{xy}\,xy + g_y x^2, \qquad (4.4.5)$$

where g_x, g_y and g_{xy} are known constants.

Assume the distribution of σ in the aperture as

$$\sigma(\rho,\phi)=\frac{a(\phi)\,[\alpha_0+\rho^2(\alpha_x\sin^2\phi+\alpha_{xy}\sin\phi\cos\phi+\alpha_y\cos^2\phi)]}{\sqrt{a^2(\phi)-\rho^2}}, \qquad (4.4.6)$$

where α_0, α_x, α_y and α_{xy} are yet unknown constants. Now it is necessary to relate α_0, α_x, α_y and α_{xy} to the parameters g_x, g_y and g_{xy}. This can be done by substitution of (6) into (4) which yields after integration with respect to ρ_0

$$w(\rho,\phi)=\alpha_0\sum_{n=-\infty}^{\infty}\int_0^{\rho}\left(\frac{x}{\rho}\right)^{|n|}\frac{x\,dx}{(\rho^2-x^2)^{1/2}}\int_0^{2\pi}e^{in(\phi-\phi_0)}F(\frac{2-|n|}{2},\frac{1}{2};1;1-\frac{x^2}{a^2(\phi_0)})\,d\phi_0$$

$$+\sum_{n=-\infty}^{\infty}\int_0^{\rho}\left(\frac{x}{\rho}\right)^{|n|}\frac{x^3\,dx}{(\rho^2-x^2)^{1/2}}\int_0^{2\pi}e^{in(\phi-\phi_0)}F\left(\frac{4-|n|}{2},\frac{1}{2};1;1-\frac{x^2}{a^2(\phi_0)}\right)$$

$$\times(\alpha_x\sin^2\phi_0+\alpha_{xy}\sin\phi_0\cos\phi_0+\alpha_y\cos^2\phi_0)\,d\phi_0\,. \qquad (4.4.7)$$

Here F stands for the Gauss hypergeometric function. Further evaluation of w can be done separately for each harmonic. Note that all the odd harmonics of w will be zero if $a(\phi)$ contains only the even harmonics. The zero*th* harmonic will take the form

$$w_0(\rho,\phi) = \frac{\pi}{4}\int_0^{2\pi} \left[2\alpha_0 + \left(a^2(\phi_0) + \frac{1}{2}\rho^2 \right) \right.$$
$$\left. \times (\alpha_x \sin^2\phi_0 + \alpha_{xy}\sin\phi_0\cos\phi_0 + \alpha_y\cos^2\phi_0) \right] a(\phi_0)\, d\phi_0, \tag{4.4.8}$$

which can be simplified as

$$w_0(\rho,\phi) = \frac{\pi}{4}\left[2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y + \frac{1}{2}\rho^2 (\alpha_x J_x + \alpha_{xy} J_{xy} + \alpha_y J_y) \right], \tag{4.4.9}$$

where the following quantities were introduced

$$B_x = \int_0^{2\pi} a^3(\phi)\sin^2\phi\, d\phi, \qquad B_y = \int_0^{2\pi} a^3(\phi)\cos^2\phi\, d\phi,$$
$$B_{xy} = \int_0^{2\pi} a^3(\phi)\sin\phi\cos\phi\, d\phi. \tag{4.4.10}$$

Since their tensor properties are similar to those of the moments of inertia, we shall call B_x and B_y *the cubic moments of a two-dimensional domain about the axes Ox and Oy* respectively, B_{xy} will be called *the cubic product of a two-dimensional domain about the axes Ox and Oy*.

$$J_0 = \int_0^{2\pi} a(\phi)\, d\phi, \qquad J_x = \int_0^{2\pi} a(\phi)\sin^2\phi\, d\phi,$$
$$J_y = \int_0^{2\pi} a(\phi)\cos^2\phi\, d\phi, \qquad J_{xy} = \int_0^{2\pi} a(\phi)\sin\phi\cos\phi\, d\phi. \tag{4.4.11}$$

These quantities were introduced in section 3.4 We shall call J_x and J_y *the linear moments of a two-dimensional domain about the axes Ox and Oy* respectively, J_{xy} will be called *the linear product of a two-dimensional domain about the axes Ox and Oy*. J_0 will be called the linear polar moment. The following property is quite clear, $J_0 = J_x + J_y$.

Here is the expression for the second harmonic,

$$w_2(\rho, \phi) = \frac{3}{8}\pi\rho^2 \int_0^{2\pi} (\alpha_x \sin^2\phi_0 + \alpha_{xy}\sin\phi_0\cos\phi_0 + \alpha_y\cos^2\phi_0)\, a(\phi_0)\cos 2(\phi - \phi_0)\, d\phi_0,$$

which can be modified as

$$w_2(\rho, \phi) = \frac{3}{8}\pi\rho^2 \{-\alpha_x[(C_{xxxx} - C_{xxyy})\cos 2\phi + 2C_{xxxy}\sin 2\phi] + \alpha_y[(C_{yyyy} - C_{xxyy})\cos 2\phi$$
$$+ 2C_{xyyy}\sin 2\phi] + \alpha_{xy}[(C_{xyyy} - C_{xxxy})\cos 2\phi + 2C_{xxyy}\sin 2\phi]\}. \tag{4.4.12}$$

Here, the following geometrical characteristics of the domain of aperture were introduced:

$$C_{xxxx} = \int_0^{2\pi} a(\phi)\sin^4\phi\, d\phi, \qquad C_{xxxy} = \int_0^{2\pi} a(\phi)\sin^3\phi\cos\phi\, d\phi,$$
$$C_{xxyy} = \int_0^{2\pi} a(\phi)\sin^2\phi\cos^2\phi\, d\phi, \qquad C_{xyyy} = \int_0^{2\pi} a(\phi)\sin\phi\cos^3\phi\, d\phi$$
$$C_{yyyy} = \int_0^{2\pi} a(\phi)\cos^4\phi\, d\phi. \tag{4.4.13}$$

The C moments will be called *the linear moments of the fourth order*. Their relationships with the J moments are easy to establish, for example, $J_x = C_{xxxx} + C_{xxyy}$, $J_{xy} = C_{xyyy} + C_{xxxy}$, *etc*. It is important to note that the parameter α_0 did not enter (12), and the parameters α_x, α_{xy}, and α_y will not enter the expression for the fourth harmonic. Investigation of the harmonics higher than 2 shows that their amplitude decreases. In the case of an ellipse they vanish thus making the solution *exact*. Of course, the odd harmonics do not vanish for the apertures of general shape, but we can always choose the system of coordinate origin in such a way as to eliminate the first harmonic and to reduce (or eliminate) the higher odd harmonics. For example, let $a(\phi) = a_1 + a_2\sin\phi$. Here we can eliminate all the odd harmonics just by moving the system of coordinate origin in the positive y direction by a_2. It can be shown that we can eliminate the first harmonic for an aperture of general shape by locating the system of coordinate origin at the center of gravity. This is why it seems justified to assume $w \approx w_0 + w_2$, ignoring the first harmonic and calling the remaining harmonics (the third and higher) the solution error. Now, we have an approximate expression for w as

$$
\begin{aligned}
w = \frac{\pi}{4}\{ & 2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y \\
& + x^2[-\alpha_x (C_{xxxx} - 2C_{xxyy}) + \alpha_{xy}(2C_{xyyy} - C_{xxxy}) + \alpha_y (2C_{yyyy} - C_{xxyy})] \\
& + y^2[\alpha_x (2C_{xxxx} - C_{xxyy}) + \alpha_{xy}(2C_{xxxy} - C_{xyyy}) - \alpha_y (C_{yyyy} - 2C_{xxyy})] \\
& + 6xy[\alpha_x C_{xxxy} + \alpha_{xy} C_{xxyy} + \alpha_y C_{xyyy}]\}.
\end{aligned} \tag{4.4.14}
$$

The comparison of (5) and (14) leads to the following set of equations:

$$
\begin{aligned}
& \frac{\pi}{4}(2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y) = 0, \\
& \frac{\pi}{4}[\alpha_x (2C_{xxxx} - C_{xxyy}) + \alpha_{xy}(2C_{xxxy} - C_{xyyy}) - \alpha_y (C_{yyyy} - 2C_{xxyy})] = g_x, \\
& \frac{\pi}{4}[-\alpha_x (C_{xxxx} - 2C_{xxyy}) + \alpha_{xy}(2C_{xyyy} - C_{xxxy}) + \alpha_y (2C_{yyyy} - C_{xxyy})] = g_y, \\
& \frac{3\pi}{2}[\alpha_x C_{xxxy} + \alpha_{xy} C_{xxyy} + \alpha_y C_{xyyy}] = g_{xy}.
\end{aligned} \tag{4.4.15}
$$

The last three equations of (15) can be solved with respect to α_x, α_y, and α_{xy}, after which the value of α_0 can be found from the first equation (15).

A significant simplification occurs when the aperture S has at least one axis of symmetry. In this case $C_{xxxy} = C_{xyyy} = B_{xy} = 0$. The last equation (15) becomes decoupled from the previous three. The solutions can be written explicitly,

$$
\begin{aligned}
& \alpha_x = \frac{4[g_x(2C_{yyyy} - C_{xxyy}) + g_y(C_{yyyy} - 2C_{xxyy})]}{3\pi (C_{xxxx} C_{yyyy} - C_{xxyy}^2)}, \\
& \alpha_y = \frac{4[g_x(C_{xxxx} - 2C_{xxyy}) + g_y(2C_{xxxx} - C_{xxyy})]}{3\pi (C_{xxxx} C_{yyyy} - C_{xxyy}^2)}, \\
& \alpha_{xy} = \frac{2g_{xy}}{3\pi C_{xxyy}}.
\end{aligned} \tag{4.4.16}
$$

Substitution of (16) into the first equation (15) gives

$$\alpha_0 = -\frac{2(g_x\beta_x + g_y\beta_y)}{3\pi J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)}, \tag{4.4.17}$$

where

$$\beta_x = B_x(2C_{yyyy} - C_{xxyy}) + B_y(C_{xxxx} - 2C_{xxyy}),$$

$$\beta_y = B_x(C_{yyyy} - 2C_{xxyy}) + B_y(2C_{xxxx} - C_{xxyy}).$$

Expressions (6), (7), (16), (17) give a complete and *exact* solution for an ellipse. We hope they will perform well for an aperture of general shape. We expect (6) to be reasonably accurate in the neighborhood of the coordinate origin while the error might become quite significant close to the boundary of the domain S, mainly due to the fact that the assumption of a square-root singularity in (6) is wrong, especially for a domain with sharp angles.

It is appropriate to discuss the following particular cases: $g_y = 2\pi/\sqrt{A}$, $g_x = g_{xy} = 0$; and the case $g_x = 2\pi/\sqrt{A}$, $g_y = g_{xy} = 0$. In every case let us compute the integral

$$p = \frac{1}{\mathrm{A}}\int\int_S \sigma\, \mathrm{dS}\,,$$

which is proportional to the average value of σ, and is dimensionless thus characterizing the shape of S and being independent of its size. We shall denote these parameters by p_y and p_x for each case respectively. These parameters correspond to the coefficients in the quadratic terms in the low-frequency expansion (Van Bladel, 1967). Formulae (7), (16), and (17) lead to the following expressions for the parameters p_y and p_x:

$$p_y = \frac{8\left(8J_0\left[I_x(C_{yyyy} - 2C_{xxyy}) + I_y(2C_{xxxx} - C_{xxyy})\right] - 3A\beta_y\right)}{9A^{3/2}J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)},$$

$$p_x = \frac{8\left(8J_0\left[I_x(2C_{yyyy} - C_{xxyy}) + I_y(C_{xxxx} - 2C_{xxyy})\right] - 3A\beta_x\right)}{9A^{3/2}J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)}, \tag{4.4.18}$$

where I_x and I_y are the well-known moments of inertia of the domain of aperture. Some further simplifications take place when the domain S possesses a central symmetry which implies that all the moments about the axis Ox are equal to the similar moments about the axis Oy. In this case

$$p_y = p_x = \frac{8(8I_0 J_0 - 3AB_0)}{3A^{3/2} J_0^2}, \tag{4.4.19}$$

where the moments with the subindex 0 indicate corresponding polar moments. Formula (17) also simplifies as follows

$$\alpha_0 = \frac{2}{\pi J_0}\left[g_0 - \frac{B_0}{J_0}(g_x + g_y)\right]. \tag{4.4.20}$$

Formulae (18) are the main results of this section. The quadratic terms in the low-frequency expansion can now be found by a relatively simple computation of the geometrical characteristics (moments) of the domain of aperture.

Several aperture shapes are considered below. The general solution is given by the formulae (18). We present only the necessary computations of the moments involved. A sufficiently high degree of accuracy of formulae derived is confirmed by comparison with available numerical solutions.

Polygon. Consider a polygon with n sides. The function $a(\phi)$ describing its boundary is bounded and single-valued. The system of coordinate origin is located at the polygon's center of gravity in order to eliminate the first harmonic from $a(\phi)$. Let us number the polygon sides in a counter-clockwise direction from 1 to n, with a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered $k+1$. It is clear that the value of index equal $n+1$ is understood as 1. Denote b_k the distance from the center of gravity to the kth apex; ψ_k stands for the angle between the axis Ox and the perpendicular to the side a_k. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. The following expressions can be obtained for the moments of inertia:

$$I_x = \sum_{k=1}^{n} -m_k \cos 2\psi_k + g_k \sin 2\psi_k + 2h_k \cos^2\psi_k ,$$

$$I_y = \sum_{k=1}^{n} m_k \cos 2\psi_k - g_k \sin 2\psi_k + 2h_k \sin^2\psi_k ,$$

$$I_{xy} = \sum_{k=1}^{n} (m_k - h_k)\sin 2\psi_k + g_k \cos 2\psi_k, \tag{4.4.21}$$

where

$$m_k = \frac{2A_k^3}{a_k^2}, \quad g_k = A_k^2 \frac{b_{k+1}^2 - b_k^2}{2a_k^2}, \quad h_k = \frac{A_k[3(b_{k+1}^2 + b_k^2) - a_k^2]}{24}. \tag{4.4.22}$$

The linear moments can be computed in the form

$$J_x = \sum_{k=1}^{n} -q_k \cos 2\psi_k + s_k \sin 2\psi_k + 2t_k \cos^2\psi_k,$$

$$J_y = \sum_{k=1}^{n} q_k \cos 2\psi_k - s_k \sin 2\psi_k + 2t_k \sin^2\psi_k, \tag{4.4.23}$$

$$J_{xy} = \sum_{k=1}^{n} (q_k - t_k)\sin 2\psi_k + s_k \cos 2\psi_k,$$

where

$$q_k = \frac{A_k}{a_k^2}\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right)[a_k^2 + (b_k - b_{k+1})^2],$$

$$s_k = 4\frac{A_k^2}{a_k^2}\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right), \tag{4.4.24}$$

$$t_k = \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}.$$

The C moments can be computed by the following formulae:

$$C_{xxxx} = \sum_{k=1}^{n} -q_k \cos 2\psi_k - u_k \cos 4\psi_k + v_k \sin 4\psi_k$$

$$+ 4s_k \sin\psi_k \cos^3\psi_k + 2t_k \cos^4\psi_k,$$

$$C_{xxxy} = \sum_{k=1}^{n} v_k \cos 4\psi_k + u_k \sin 4\psi_k + s_k \cos^2\psi_k \times (1 - 4\sin^2\psi_k) + \tfrac{1}{2} q_k \sin 2\psi_k - 2t_k \sin\psi_k \cos^3\psi_k,$$

$$C_{xxyy} = \sum_{k=1}^{n} u_k \cos 4\psi_k - v_k \sin 4\psi_k - \tfrac{1}{2} s_k \sin 4\psi_k + 2t_k \sin^2\psi_k \cos^2\psi_k,$$

$$C_{xyyy} = \sum_{k=1}^{n} -v_k \cos 4\psi_k - u_k \sin 4\psi_k - s_k \sin^2\psi_k \times (1 - 4\cos^2\psi_k) + \tfrac{1}{2} q_k \sin 2\psi_k - 2t_k \sin^3\psi_k \cos\psi_k,$$

$$C_{yyyy} = \sum_{k=1}^{n} q_k \cos 2\psi_k - u_k \cos 4\psi_k + v_k \sin 4\psi_k - 4s_k \sin^3\psi_k \cos\psi_k + 2t_k \sin^4\psi_k,$$

where q_k, s_k and t_k are defined by (24) and

$$u_k = \frac{A_k}{12a_k^4}\left\{\left[\frac{b_{k+1}^2 - b_k^2 + a_k^2}{b_{k+1}}\right]^3 + \left[\frac{b_k^2 - b_{k+1}^2 + a_k^2}{b_k}\right]^3\right\}, \tag{4.4.25}$$

$$v_k = \frac{16A_k^4}{3a_k^4}\left[\frac{1}{b_{k+1}^3} - \frac{1}{b_k^3}\right].$$

The following formulae can be derived for the cubic moments:

$$B_x = \sum_{k=1}^{n} -j_k \cos 2\psi_k + r_k \sin 2\psi_k + 2f_k \cos^2\psi_k,$$

$$B_y = \sum_{k=1}^{n} j_k \cos 2\psi_k - r_k \sin 2\psi_k + 2f_k \sin^2\psi_k, \tag{4.4.26}$$

$$B_{xy} = \sum_{k=1}^{n} (j_k - f_k) \sin 2\psi_k + r_k \cos 2\psi_k,$$

where

$$j_k = \left(\frac{2A_k}{a_k}\right)^3 \ln\frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k},$$

$$r_k = \left(\frac{2A_k}{a_k}\right)^2 (b_{k+1} - b_k),$$

$$f_k = \tfrac{1}{4}(b_k + b_{k+1})A_k\left[1 + \left(\frac{b_{k+1} - b_k}{a_k}\right)^2\right] + \tfrac{1}{4} j_k.$$

Substitution of (21–26) into (18) gives the complete solution for an arbitrary polygon. In the case of a regular polygon $a_k = a$, $b_k = b = a/[2\sin(\pi/n)]$, $\psi_k = 2\pi(k - 1)/n$, $A_k = [a^2\cot(\pi/n)]/4 = [b^2\sin(2\pi/n)]/2$, $A = nA_k$, and formulae (21–26) simplify to

$$I_x = I_y = \frac{na^4}{64}\cot\frac{\pi}{n}\left[\cot^2\frac{\pi}{n} + \frac{1}{3}\right] = \frac{nb^4}{24}\sin\frac{2\pi}{n}\left[2 + \cos\frac{2\pi}{n}\right], \tag{4.4.27}$$

$$J_x = J_y = \tfrac{1}{4} na\cot\frac{\pi}{n}\ln\frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)} = \tfrac{1}{2} nb\cos\frac{\pi}{n}\ln\frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}, \tag{4.4.28}$$

$$B_x = B_y = \tfrac{1}{4} nb^3\left[\sin\frac{2\pi}{n} + \cos^3\frac{\pi}{n}\ln\frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}\right], \tag{4.4.29}$$

$$C_{xxxx} = C_{yyyy} = \tfrac{3}{8} nb\cos\frac{\pi}{n}\ln\frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)},$$

$$C_{xxyy} = \tfrac{1}{8} nb\cos\frac{\pi}{n}\ln\frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}. \tag{4.4.30}$$

Note that formulae (30) are valid for any regular polygon except the square, due to the fact that the trigonometric series summation, namely,

$$\sum_{k=1}^{n} \sin^4\left[(k\text{-}1)\frac{2\pi}{n}\right] = \sum_{k=1}^{n} \cos^4\left[(k\text{-}1)\frac{2\pi}{n}\right] = \tfrac{3}{8} n,$$

is not valid for a square. The C moments for a square with the side equal $2l$ can be expressed as

$$C_{xxxx} = C_{yyyy} = l\left[4\ln(1+\sqrt{2}) - \frac{2\sqrt{2}}{3}\right],$$
$$C_{xxyy} = \frac{2\sqrt{2}}{3}l. \tag{4.4.31}$$

Formulae (6, 7, 16, and 17) simplify for a regular polygon

$$\alpha_x = \frac{4(5g_x + g_y)}{3\pi J_0},$$
$$\alpha_y = \frac{4(g_x + 5g_y)}{3\pi J_0}, \tag{4.4.32}$$
$$\alpha_{xy} = \frac{16g_{xy}}{3\pi J_0}.$$

Again, one should note that formulae (32) are not valid for a square. The formulae to follow are valid for an arbitrary polygon including the square.

$$\alpha_0 = -\frac{2B_0}{\pi J_0{}^2}(g_x + g_y).$$

The dimensionless coefficients p_y and p_x will take the form

$$p_y = p_x = \frac{8\sqrt{2}}{\left(n\sin\frac{2\pi}{n}\right)^{1/2} n\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}} \times\left[\frac{23+7\cos\frac{2\pi}{n}}{36} - \frac{\sin\frac{\pi}{n}}{\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}}\right]. \tag{4.4.33}$$

Consider several particular values of n. For an equilateral triangle ($n = 3$) formula (33) gives $p_y = p_x = 0.3782$. We did not find any numerical data to compare with this result. In the case of a square $n = 4$, and $p_y = p_x = 0.2697$. The result due to De Smedt (1979) is 0.2645, with the discrepancy less than 2%. Since formula (33), in the limiting case $n \to \infty$, gives the exact result for a circle $p_y = p_x = 4/(3\pi^{3/2}) =$ 0.2394, we should expect that the error of (33) will decrease with n. The value of the coefficients for a regular hexagon is 0.2443, and again, we did not find anything in the literature to compare with this result. It is noteworthy that the value of the coefficients does not change significantly in the whole range $3 \le n < \infty$.

Rectangle. Consider an aperture with a rectangular base, a_1 and a_2 being its semiaxes along the axis Ox and Oy respectively. Introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (24–29) in this case reduce to

$$I_x = (4/3)\, a_1 a_2^3, \qquad I_y = (4/3)\, a_1^3 a_2\,, \tag{4.4.34}$$

$$J_x = 4a_1 \sinh^{-1} \varepsilon\,, \qquad J_y = 4a_2 \sinh^{-1}(1/\varepsilon), \tag{4.4.35}$$

$$C_{xxxx} = 4a_1\left(\sinh^{-1} \varepsilon - \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}\right),$$

$$C_{xxyy} = 4a_1 \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}, \tag{4.4.36}$$

$$C_{yyyy} = 4a_2\left(\sinh^{-1} \frac{1}{\varepsilon} - \frac{1}{3\sqrt{1+\varepsilon^2}}\right),$$

$$B_x = 2a_1^3\left(\varepsilon\sqrt{1+\varepsilon^2} - \sinh^{-1} \varepsilon + 2\varepsilon^3 \sinh^{-1} \frac{1}{\varepsilon}\right),$$

$$B_y = 2a_1^3\left[\varepsilon\sqrt{1+\varepsilon^2} - \varepsilon^3 \sinh^{-1} \frac{1}{\varepsilon} + 2\sinh^{-1} \varepsilon \right]. \tag{4.4.37}$$

We have found in the literature some numerical results which seem to be more or less accurate. The coefficients p_y and p_x were computed by De Smedt (1979) for a rectangle with various aspect ratio ε. Here, we present his results along with those given by the method of this section

ε=	0.1000	0.2000	0.3330	0.5000	0.7500	1.0000
De Smedt p_y=	2.9980	1.3730	0.7942	0.5229	0.3491	0.2645
our result p_y=	3.2809	1.3959	0.7782	0.5100	0.3485	0.2697
Discrepancy in p_y %	-9.4	-1.7	2.0	2.5	0.2	-2.0
De Smedt p_x=	0.0376	0.0639	0.0982	0.1399	0.2022	0.2645
our result p_x=	0.0284	0.0577	0.0963	0.1431	0.2086	0.2697
Discrepancy in p_x %	24.6	9.7	1.9	-2.3	-3.2	-2.0

Our formulae seem to perform satisfactorily in a sufficiently wide range of aspect ratio. The distribution of σ due to (7) can be compared with the numerical data received in a personal communication from De Smedt. Computations were made for $\varepsilon = 0.5$, $g_y = 2\pi/\sqrt{A}$, $g_x = 0$. Here are the results along the axis Ox, compared to those communicated by De Smedt

$x/a_1=$	0.0000	0.0833	0.2500	0.3333	0.5000	0.6667	0.7500	0.9167
De Smedt $\sigma=$	-0.4715	-0.4673	-0.3933	-0.3249	-0.1238	0.2515	0.5456	2.0580
our result $\sigma=$	-0.4731	-0.4647	-0.3953	-0.3314	-0.1290	0.2273	0.5141	1.8556
Discrepancy %	-0.3	0.6	-0.5	-2.0	-4.2	9.6	-1.5	9.8

We compare the same values along the axis Oy.

$y/a_2=$	0.0000	0.1667	0.3333	.5000	.6667	.8333
De Smedt $\sigma=$	-0.4715	-0.4765	-0.4774	-0.4837	-0.5063	-0.5311
our result $\sigma=$	-0.4731	-0.4744	-0.4791	-0.4907	-0.5198	-0.6138
Discrepancy %	-0.3	0.5	-0.3	-1.4	-2.7	-15.6

As we predicted, the discrepancy becomes quite significant close to the boundary.

Rhombus. Let a_1 and a_2 be its semiaxes along Ox and Oy respectively. Denote its side $l = (a_1^2 + a_2^2)^{1/2}$, and introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (21–26) in this case yield

$$I_x = \frac{l^4\varepsilon^3}{3(1+\varepsilon^2)^2}, \qquad I_y = \frac{l^4\varepsilon}{3(1+\varepsilon^2)^2}, \qquad A = \frac{2l^2\varepsilon}{(1+\varepsilon^2)}. \tag{4.4.38}$$

$$J_x = \frac{4l\varepsilon}{(1+\varepsilon^2)}\left[\frac{1-\varepsilon}{\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$J_y = \frac{4l\varepsilon}{(1+\varepsilon^2)}\left[-\frac{1-\varepsilon}{\sqrt{1+\varepsilon^2}} + \frac{1}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right]. \tag{4.4.39}$$

$$B_x = \frac{2l^3\varepsilon^3}{(1+\varepsilon^2)^3}\left[\frac{\varepsilon^3+4\varepsilon-3}{\sqrt{1+\varepsilon^2}} + \frac{2-\varepsilon^2}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$B_y = \frac{2l^3\varepsilon}{(1+\varepsilon^2)^3}\left[\frac{1+4\varepsilon^2-3\varepsilon^3}{\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2(2\varepsilon^2-1)}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right]. \tag{4.4.40}$$

$$C_{xxxx} = \frac{4l\varepsilon}{(1+\varepsilon^2)^2}\left[\frac{2-\varepsilon+5\varepsilon^2-4\varepsilon^3}{3\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^4}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$C_{yyyy} = \frac{4l\varepsilon}{(1+\varepsilon^2)^2}\left[\frac{-4+5\varepsilon-\varepsilon^2+2\varepsilon^3}{3\sqrt{1+\varepsilon^2}} + \frac{1}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$C_{xxyy} = \frac{4l\varepsilon}{(1+\varepsilon^2)^2}\left[\frac{1-2\varepsilon-2\varepsilon^2+\varepsilon^3}{3\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right].$$

Again, we have only the numerical results by De Smedt (1979) to compare with ours which are given below

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt p_y=	4.6520	1.8890	0.9844	0.5933	0.3655	0.2631
our result p_y=	3.7425	1.6605	0.9192	0.5770	0.3661	0.2697
Discrepancy %	19.6	12.1	6.6	2.7	-0.2	-2.5
De Smedt p_x=	0.0314	0.0549	0.0862	0.1270	0.1923	0.2631
our result p_x=	0.1944	0.1435	0.1345	0.1532	0.2050	0.2697
Discrepancy %	-518.4	-161.3	-56.0	-20.6	-6.6	-2.5

Though our results are satisfactory for p_y, they are unacceptable for p_x when $\varepsilon \le 0.5$. The reason for this is our assumption of a square-root singularity in (6) which is grossly incorrect for contours with sharp angles. An alternative approach which uses the variational principle and somewhat improves the accuracy, is discussed further.

Cross. Consider an aperture with a configuration obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$ $(a \ge b)$. Introduce the aspect ratio as $\varepsilon = b/a$. The area and the moments will take the form

$$A = 4a^2\varepsilon(2-\varepsilon), \qquad I_x = I_y = \frac{4}{3}a^4\varepsilon(1+\varepsilon^2-\varepsilon^3),$$

$$J_x = J_y = 4a\left[\ln(\varepsilon+\sqrt{1+\varepsilon^2}) + \varepsilon\ln\frac{1+\sqrt{1+\varepsilon^2}}{(1+\sqrt{2})\varepsilon}\right], \tag{4.4.41}$$

$$B_x = B_y = 2a^3\left\{2\varepsilon\sqrt{1+\varepsilon^2} + \ln(\varepsilon+\sqrt{1+\varepsilon^2}) + \varepsilon^3\left[\ln\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon(1+\sqrt{2})} - \sqrt{2}\right]\right\}.$$

The comparison between the results of this section and those given by De Smedt (1979) are presented below

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt p_y=p_x=	0.9675	0.4854	0.3271	0.2671	0.2523	0.2645
our result p_y=p_x=	1.6943	0.6765	0.3716	0.2683	0.2517	0.2697
Discrepancy %	-75.1	-39.4	-13.6	-0.5	0.3	-2.0

Taking into consideration the shape complexity, we should consider the resulting agreement as surprisingly good, not only quantitatively but qualitatively as well: both data display a relatively flat minimum around $\varepsilon = 0.75$. The discrepancy becomes unacceptably big for $\varepsilon \leq 0.3$. It will be shown further that the variational approach slightly improves the results.

Variational approach. An alternative method can be suggested by using the variational approach (Noble 1960). The following functional assumes its stationary value at the exact solution of (1)

$$I(\sigma) = 2\iint_S \sigma(M)w(M)\mathrm{d}S_M - \iint_S \sigma(M)\left[\iint_S \frac{\sigma(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \tag{4.4.42}$$

Take

$$\iint_S \frac{\sigma(N)}{R(M,N)}\mathrm{d}S_N \approx w_0 + w_2, \tag{4.4.43}$$

where σ is defined by (6) and $w_0 + w_2$ is given by (14). Substitution of (5), (6), (14), and (43) into (42) makes it possible to consider the functional I as a function of α_0, α_x, α_y, and α_{xy}. The extremum conditions

$$\frac{\partial I}{\partial\alpha_0} = 0, \quad \frac{\partial I}{\partial\alpha_x} = 0, \quad \frac{\partial I}{\partial\alpha_y} = 0, \quad \frac{\partial I}{\partial\alpha_{xy}} = 0,$$

give four linear algebraic equations with respect to the unknown α_0, α_x, α_y, and α_{xy}. The complete solution is pretty cumbersome. Here, we present the set of equations for the coefficients α_0, α_x, and α_y which are valid only for the domains having at least one axis of symmetry.

$$\begin{aligned} c_{11}\alpha_0 + c_{12}\alpha_x + c_{13}\alpha_y &= \frac{16}{3}(I_x g_x + I_y g_y), \\ c_{12}\alpha_0 + c_{22}\alpha_x + c_{23}\alpha_y &= \frac{16}{15}(D_{xxxx} g_x + D_{xxyy} g_y), \\ c_{13}\alpha_0 + c_{23}\alpha_x + c_{33}\alpha_y &= \frac{16}{15}(D_{xxyy} g_x + D_{yyyy} g_y). \end{aligned} \tag{4.4.44}$$

Here,

$$c_{11} = 2\pi J_0 A,$$

$$c_{12} = \frac{1}{2}\pi\left[B_x A + \frac{4}{3}I_x(2J_0 + 2C_{xxxx} - C_{xxyy}) - \frac{4}{3}I_y(C_{xxxx} - 2C_{xxyy})\right],$$

$$c_{13} = \frac{1}{2}\pi\,[B_y A + \frac{4}{3} I_y (2J_0 + 2C_{yyyy} - C_{xxyy}) - \frac{4}{3} I_x (C_{yyyy} - 2C_{xxyy})],$$

$$c_{22} = \frac{4}{15}\pi\,[5B_x I_x + D_{xxxx}(2C_{xxxx} - C_{xxyy}) - D_{xxyy}(C_{xxxx} - 2C_{xxyy})],$$

$$c_{23} = \frac{2}{15}\pi\,[5(B_x I_y + B_y I_x) - D_{xxxx}(C_{yyyy} - 2C_{xxyy}) + 2D_{xxyy}(C_{xxxx} + C_{yyyy} - C_{xxyy}) - D_{yyyy}(C_{xxxx} - 2C_{xxyy})],$$

$$c_{33} = \frac{4}{15}\pi\,[5B_y I_y + D_{yyyy}(2C_{yyyy} - C_{xxyy}) - D_{xxyy}(C_{yyyy} - 2C_{xxyy})]. \qquad (4.4.45)$$

The D moments are introduced similar to (13) as

$$D_{xxxx} = \int_0^{2\pi} a^6(\phi)\sin^4\phi\, d\phi\,, \qquad D_{xxyy} = \int_0^{2\pi} a^6(\phi)\sin^2\phi\cos^2\phi\, d\phi\,,$$

$$D_{yyyy} = \int_0^{2\pi} a^6(\phi)\cos^4\phi\, d\phi. \qquad (4.4.46)$$

It is quite clear that the variational approach solution is more cumbersome than the one introduced in the first part. It remains to be seen whether it will be more accurate. One advantage should be noted: the matrix of (44) is symmetric (as it is required by the reciprocal theorem) while the matrix of (15) generally is not symmetric.

Let us compare the results for several particular configurations. First of all, consider a regular polygon. In the tables hereafter the word *simple* refers to the method introduced in (18), the word *variational* refers to the solution of the set of equations (44). In the case of a regular polygon we shall need the polar D moment only

$$D_0 = \tfrac{1}{5} b^6 n \sin\frac{2\pi}{n}\left[\,1 + \tfrac{4}{3}\cos^2\frac{\pi}{n} + \tfrac{8}{3}\cos^4\frac{\pi}{n}\right].$$

Here are the results of computations for a regular polygon with n sides

$n=$	3	4	5	6	7	9	100
simple $p_y=p_x=$	0.3782	0.2697	0.2502	0.2443	0.2420	0.2403	0.2394
variational $p_y=p_x=$	0.3409	0.2612	0.2472	0.2429	0.2412	0.2401	0.2394
Discrepancy %	9.9	3.2	1.2	0.6	0.3	0.1	0.0

Both methods seem to work well. If one considers the result by De Smedt (1979) for a square 0.2645 as exact then this might be an indication that the variational approach is somewhat more accurate. In the limiting case of $n \to \infty$ both methods give the exact result for a circle.

The D moments for the rectangle considered earlier will take the form

$$D_{xxxx} = \frac{24}{5} a_1 a_2^5, \quad D_{yyyy} = \frac{24}{5} a_1^5 a_2, \quad D_{xxyy} = \frac{8}{3} a_1^3 a_2^3.$$

Here are the numerical results computed for a rectangle

$\varepsilon=$	0.1000	0.2000	0.3330	0.5000	0.7500	1.0000
De Smedt $p_y=$	2.9980	1.3730	0.7942	0.5229	0.3491	0.2645
variational $p_y=$	3.4239	1.4523	0.8023	0.5166	0.3437	0.2612
Discrepancy %	-14.2	-5.8	-1.0	1.2	1.5	1.2
De Smedt $p_x=$	0.0376	0.0639	0.0982	0.1399	0.2022	0.2645
variational $p_x=$	0.0316	0.0588	0.0939	0.1370	0.1997	0.2612
Discrepancy %	15.9	7.9	4.3	2.1	1.2	1.2

Again, the general impression is that the variational approach is more accurate but not everywhere, for example, the discrepancy in p_y for $\varepsilon = 0.1$ increased as compared to the simple method result given earlier.

The D moments for a rhombus will take the form

$$D_{xxxx} = \frac{4}{5} a_1 a_2^5, \quad D_{yyyy} = \frac{4}{5} a_1^5 a_2, \quad D_{xxyy} = \frac{2}{15} a_1^3 a_2^3.$$

We present below the numerical results for a rhombus due to the variational approach compared to those by De Smedt (1979)

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y=$	4.6520	1.8890	0.9844	0.5933	0.3655	0.2631
variational $p_y=$	-0.5952	3.3549	0.9465	0.5580	0.3534	0.2612
Discrepancy %	112.8	-77.6	3.8	6.0	3.3	0.7
De Smedt $p_x=$	0.0314	0.0549	0.0862	0.1270	0.1923	0.2631
variational $p_x=$	0.0090	0.1464	0.1110	0.1400	0.1971	0.2612
Discrepancy %	71.5	-166.7	-28.8	-10.2	-2.5	0.7

Though the discrepancy decreased for $\varepsilon > 0.33$, we should state that both methods fail for a domain with sharp angles, since the results are unacceptable for $\varepsilon < 0.33$.

In the case of a cross-shaped aperture, the D moments can be expressed as follows:

$$D_{xxxx} = D_{yyyy} = \frac{24}{5} l^6 \varepsilon (1 + \varepsilon^4 - \varepsilon^5), \quad D_{xxyy} = \frac{8}{3} l^6 \varepsilon^3 (2 - \varepsilon^3).$$

Here are the numerical results due to the variational approach compared to those by De Smedt (1979)

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y=p_x=$	0.9675	0.4854	0.3271	0.2671	0.2523	0.2645
variational $p_y=p_x=$	1.4346	0.5822	0.3397	0.2606	0.2482	0.2612
Discrepancy %	-48.3	-19.9	-3.9	2.4	1.6	1.2

Comparison of this table with a similar one given earlier leads to the same conclusion: the results become valid in a wider range of the aspect ratio ε, but the theory fails for very small ε. It is up to the user to decide whether a somewhat better accuracy of the variational approach is worth more cumbersome computations.

We have to caution the reader willing to use the reciprocal theorem and the solution in sections 3.3 and 3.4 in order to find further terms in the low-frequency expansion. The results might be good for the domains with the aspect ratio close to unity (like, for example, a square) but the accuracy deteriorates quickly as the aspect ratio moves away from unity. It is advisable in each particular case to use the method similar to the one in this section.

Formulae (18) give a simple and effective solution to the problem of evaluating the quadratic terms in the low-frequency expansion for the problem of sound penetration through an aperture in a rigid screen. Their high accuracy in a sufficiently wide range of aspect ratio is confirmed by numerous examples. The case of a domain with sharp angles seems to be outside this class. An investigation of the nature of singularity is absolutely indispensable for this type of problems. A similar method can be used for evaluating further terms of the low-frequency expansion.

4.5. Sound penetration through a general aperture in a soft screen

The term soft screen represents an abstraction opposite to that of a rigid screen. The diffraction of a plane wave by an aperture in a soft screen is an important acoustical problem. Again, very little is known about the apertures of general shape, except for some numerical solutions (De Meulenaere and Van Bladel, 1977; Okon and Harrington, 1981). Here we reproduce some essential results from (Van Bladel, 1968) which are necessary for better understanding of the problem formulation.

Consider a flat soft screen with a general aperture S, whose boundary is given in the polar coordinates as

$$\rho = a(\phi). \tag{4.5.1}$$

Let the incident field be a plane wave $P^i = e^{-jkR}$, where $R = \mathbf{u}_i \cdot \mathbf{r}$, $\mathbf{u}_i$ is the incidence vector and $\mathbf{r}$ is the field point vector and k is the wave number. The governing integral equation in the case of a soft screen takes the form

$$\frac{1}{2\pi} \int\int_S P(M) \frac{\partial}{\partial z_M} \left[\frac{e^{-jkR(M,N)}}{R(M,N)} \right] dS_M = P^i(N),$$

where P is the acoustic pressure in the aperture. In the low-frequency case the characteristic length of the aperture is much smaller that the wavelength, and the following expansion becomes valid

$$P(\mathbf{r}) = P_0(\mathbf{r}) + jkP_1(\mathbf{r}) + \frac{1}{2}(jk)^2 P_2(\mathbf{r}) + \dots ,$$

Van Bladel (1968) has proven that the diffraction problem can be reduced to the solution of a sequence of integral equations of the following type

$$\sigma(N) = \Delta \int\int_S \frac{w(M)}{R(M,N)} \, dS, \tag{4.5.2}$$

where Δ is the two-dimensional Laplace operator, S is the aperture domain, $R(M,N)$ stands for the distance between the points M and N, w denotes the unknown function and σ is a known function, If we denote w°, w_x, w_y, w_{xx}, *etc.* as solutions of (2) corresponding respectively to the function σ taking on values $-2\pi/\sqrt{A}$, $-2\pi x/A$, $-2\pi y/A$, $-2\pi x^2/A^{3/2}$, *etc.*, where A is the area of the aperture, then the various parameters can be defined quite simply through these solutions. For example,

$$P_1 = jk\sqrt{A} \cos\theta_i \, w^{\circ},$$

where θ_i is the angle of incidence. The reader is referred to the original paper by Van Bladel (1968) for the rest of the theory. The most important seems to be the zeroth-order term w°. The analytical solution w° is known for a circle and an ellipse only. The case of non-elliptic aperture had to be treated numerically. The problem of sound penetration through an aperture is mathematically equivalent to the one of the electrical polarizability. Therefore, in order to avoid unnecessary repetition, the reader is referred for the rest of the theory to section 3.5.

CHAPTER 5

NEW SOLUTIONS IN CONTACT MECHANICS

This Chapter contains complete solutions to several contact problems which were obtained recently, and could not be included in (Fabrikant, 1989a). Those comprise complete elastic fields around axisymmetric and inclined bonded punch. These fundamental solutions allow us to solve various problems of interaction between punches and anchor loads. Two of such solutions are included. A new approach is presented to a general annular punch problem, with analytical, numerical and asymptotic solutions derived and compared.

5.1. Axisymmetric bonded punch problem

The bonded punch problem belongs to the class of the mixed-mixed problems of elasticity theory which are among the most complicated due to the coupling between the normal and tangential parameters. We should mention the works of Mossakovskii (1954) and Ufliand (1956) among the first published exact solutions for the case of an *isotropic* half-space, obtained by using various integral transforms. A more compact solution has been reported by Kapshivyi and Masliuk (1967), who used a special apparatus of *p*-analytical functions. The first *elementary* exact solution for a *transversely isotropic* elastic half-space was published in (Fabrikant, 1971c). Four different types of solution of the governing set of integral equations were reported in (Fabrikant, 1986b).

All these solutions define the elastic field in the plane z=0 only. We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic half-space. One may argue that since the stresses exerted at the punch base are known, we can substitute them into the Boussinesq point force solution (which is well known, for example, see Fabrikant, 1970) and obtain the complete solution in quadratures. Theoretically, yes, this can be done, but practically, this solution would be of little use since it would require triple integration, with one being singular, and a numerical differentiation. The computing time for this procedure would be quite significant, and its accuracy would be very doubtful. This is the main reason why, to the best of our knowledge, nobody tried so far to obtain a complete solution, even in the case of an isotropic body. On the other hand, knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems of interaction between a bonded punch and anchor loads or cracks.

The complete solution has become possible due to the new results presented in Chapter 1. The expressions for the stresses exerted by the punch are fed in the point force solution, with one important distinction: two of the three integrations and the differentiation are performed exactly, and lead to remarkably simple and elementary expressions involving only one non-singular integration. The case of a circular centrally loaded punch bonded to an elastic half-space is considered as an example. Numerical results are obtained in order to compare the field of normal and tangential displacements due to a bonded punch with similar results for a smooth punch.

Theory. Consider a transversely isotropic elastic body which is characterized by five elastic constants A_{ik} defining the following stress-strain relationships:

$$\sigma_x = A_{11}\frac{\partial u_x}{\partial x} + (A_{11} - 2A_{66})\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_y = (A_{11} - 2A_{66})\frac{\partial u_x}{\partial x} + A_{11}\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z = A_{13}\frac{\partial u_x}{\partial x} + A_{13}\frac{\partial u_y}{\partial y} + A_{33}\frac{\partial w}{\partial z},$$

$$\tau_{xy} = A_{66}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}), \quad \tau_{yz} = A_{44}(\frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}),$$

$$\tau_{zx} = A_{44}(\frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z}). \qquad (5.1.1)$$

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \qquad (5.1.2)$$

Substitution of (1) in (2) yields:

$$A_{11}\frac{\partial^2 u_x}{\partial x^2}+A_{66}\frac{\partial^2 u_x}{\partial y^2}+A_{44}\frac{\partial^2 u_x}{\partial z^2}+(A_{11}-A_{66})\frac{\partial^2 u_y}{\partial x\partial y}+(A_{13}+A_{44})\frac{\partial^2 w}{\partial x\partial z}=0\,,$$

$$A_{66}\frac{\partial^2 u_y}{\partial x^2}+A_{11}\frac{\partial^2 u_y}{\partial y^2}+A_{44}\frac{\partial^2 u_y}{\partial z^2}+(A_{11}-A_{66})\frac{\partial^2 u_x}{\partial x\partial y}+(A_{13}+A_{44})\frac{\partial^2 w}{\partial y\partial z}=0\,,$$

$$A_{44}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]+A_{33}\frac{\partial^2 w}{\partial z^2}+(A_{44}+A_{13})\left[\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}\right]=0\,. \tag{5.1.3}$$

Introduce complex tangential displacements $u=u_x+iu_y$, and $\bar{u}=u_x-iu_y$. This will allow us to reduce the number of equations in (3) by one, and to rewrite these equations in a more compact manner, namely,

$$\frac{1}{2}(A_{11}+A_{66})\Delta u+A_{44}\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}(A_{11}-A_{66})\Lambda^2\bar{u}+(A_{13}+A_{44})\Lambda\frac{\partial w}{\partial z}=0,$$

$$A_{44}\Delta w+A_{33}\frac{\partial^2 w}{\partial z^2}+\frac{1}{2}(A_{13}+A_{44})\frac{\partial}{\partial z}(\bar{\Lambda}u+\Lambda\bar{u})=0. \tag{5.1.4}$$

Here the following differential operators were used:

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\,,\qquad \Lambda=\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\,, \tag{5.1.5}$$

and the overbar everywhere indicates the complex conjugate value. Note also that $\Delta=\Lambda\bar{\Lambda}$. One can verify that equations (4) can be satisfied by

$$u=\Lambda(F_1+F_2+iF_3),\qquad w=m_1\frac{\partial F_1}{\partial z}+m_2\frac{\partial F_2}{\partial z}, \tag{5.1.6}$$

where all three functions F_k satisfy the equation (Elliott, 1948):

$$\Delta F_k+\gamma_k^2\frac{\partial^2 F_k}{\partial z^2}=0,\qquad\qquad \text{for } k=1,2,3, \tag{5.1.7}$$

and the values of m_k and γ_k are related by the following expressions (Elliott, 1948):

$$\frac{A_{44}+m_k(A_{13}+A_{44})}{A_{11}}=\frac{m_kA_{33}}{m_kA_{44}+A_{13}+A_{44}}=\gamma_k^2,\quad \text{for } k=1,2;$$

$$\gamma_3=\left(A_{44}/A_{66}\right)^{1/2}. \tag{5.1.8}$$

Introducing the notation $z_k=z/\gamma_k$, for k=1,2,3, we may call function $F_k=F(x,y,z_k)$ harmonic. Note the property $m_1m_2=1$, which seems to have escaped the attention of previous researchers, and which will help us to simplify various expressions to follow. The other elastic constants which will be used throughout the section are:

$$G_1=\beta+\gamma_1\gamma_2H,\qquad G_2=\beta-\gamma_1\gamma_2H,$$

$$H=\frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)},\quad \alpha=\frac{(A_{11}A_{33})^{1/2}-A_{13}}{A_{11}(\gamma_1+\gamma_2)},\quad \beta=\frac{\gamma_3}{2\pi A_{44}}. \tag{5.1.9}$$

Introduce the following inplane stress components:

$$\sigma_1=\sigma_x+\sigma_y,\quad \sigma_2=\sigma_x-\sigma_y+2i\tau_{xy},\quad \tau_z=\tau_{zx}+i\tau_{yz}. \tag{5.1.10}$$

This will simplify expressions (1), namely

$$\sigma_1=(A_{11}-A_{66})(\bar{\Lambda}u+\Lambda\bar{u})+2A_{13}\frac{\partial w}{\partial z},\quad \sigma_2=2A_{66}\Lambda u,$$

$$\sigma_z=\frac{1}{2}A_{13}(\bar{\Lambda}u+\Lambda\bar{u})+A_{33}\frac{\partial w}{\partial z},\quad \tau_z=A_{44}\left[\frac{\partial u}{\partial z}+\Lambda w\right]. \tag{5.1.11}$$

We have now only four components of stress, instead of six, as it was in (1). The substitution of (6) in (11) yields:

$$\sigma_1=2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2-(1+m_1)\gamma_3^2]F_1+[\gamma_2^2-(1+m_2)\gamma_3^2]F_2\},$$

$$\sigma_2=2A_{66}\Lambda^2(F_1+F_2+iF_3),$$

$$\sigma_z=A_{44}\frac{\partial^2}{\partial z^2}[(1+m_1)\gamma_1^2F_1+(1+m_2)\gamma_2^2F_2]$$

$$= -A_{44}\,\Delta[(1+m_1)F_1 + (1+m_2)F_2]\,,$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1 + (1+m_2)F_2 + iF_3]\,. \qquad (5.1.12)$$

Here we used the fact that each F_k satisfies equation (7), and the relation: $A_{11}\gamma_k^2 - A_{13}m_k = A_{44}(1+m_k)$, (for k=1,2) which is an immediate consequence of (8). Expressions (6) and (12) give a general solution, expressed in terms of three harmonic functions F_k. It is very attractive to express each function F_k through just *one* harmonic function as follows:

$$F_k(x,y,z) = c_k F(x,y,z_k)\,, \qquad (5.1.13)$$

where $z_k = z/\gamma_k$, and c_k is an as yet unknown complex constant. As we shall see further, this is possible indeed. All the results obtained in this section are valid for isotropic solids, provided that we take

$$\gamma_1 = \gamma_2 = \gamma_3 = 1, \quad H = \frac{1-\nu^2}{\pi E}, \quad \alpha = \frac{1-2\nu}{2(1-\nu)},$$

$$\beta = \frac{1+\nu}{\pi E}, \quad G_1 = \frac{(2-\nu)(1+\nu)}{\pi E}, \quad G_2 = \frac{\nu(1+\nu)}{\pi E}, \qquad (5.1.14)$$

where E is the elastic modulus, and ν is Poisson coefficient.

Consider a transversely isotropic elastic half-space $z \geq 0$. Let a point force, with components T_x, T_y, and P in Cartesian coordinates be applied at the point N_0 located at the boundary z=0 of a transversely isotropic elastic half-space. We may assume, without loss of generality, that the polar cylindrical coordinates of N_0 are $(\rho_0,\phi_0,0)$. We need to find the field of stresses and displacements at the point $M(\rho,\phi,z)$. Introduce the complex tangential force $T = T_x + iT_y$. The general solution can be expressed through the three potential functions:

$$F_1 = \frac{H\gamma_1}{m_1-1}\left[\frac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1+z_1)\right],$$

$$F_2 = \frac{H\gamma_2}{m_2-1}\left[\frac{1}{2}\gamma_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2+z_2)\right],$$

$$F_3 = \mathrm{i}\,\frac{\gamma_3}{4\pi A_{44}}(\bar{\Lambda}\chi_3 - \Lambda\bar{\chi}_3). \qquad (5.1.15)$$

Here

$$\chi_k(z) = \chi(z_k), \quad R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_k^2]^{1/2}, \quad \text{for} \quad k=1,2,3;$$

$$\chi(z) = T[z\ln(R_0 + z) - R_0], \quad R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}. \tag{5.1.16}$$

Substitution of (15–16) in (6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\overline{T}}{R_3(R_3+z_3)^2}\right]$$

$$+ \frac{H\gamma_2}{m_2-1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2} + \frac{q^2\overline{T}}{R_2(R_2+z_2)^2}\right] + \frac{Pq}{R_2(R_2+z_2)}\right\}$$

$$+ \frac{H\gamma_1}{m_1-1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1} + \frac{q^2\overline{T}}{R_1(R_1+z_1)^2}\right] + \frac{Pq}{R_1(R_1+z_1)}\right\}, \tag{5.1.17}$$

$$w = H\left\{\frac{1}{2}(T\overline{q} + \overline{T}q)\left[\frac{\gamma_2 m_1}{(m_1-1)R_1(R_1+z_1)} + \frac{\gamma_1 m_2}{(m_2-1)R_2(R_2+z_2)}\right]\right.$$

$$\left. + P\left[\frac{m_1}{(m_1-1)R_1} + \frac{m_2}{(m_2-1)R_2}\right]\right\}. \tag{5.1.18}$$

Here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \tag{5.1.19}$$

Expressions (17) and (18) simplify for the case when z=0

$$u = \frac{1}{2}G_1\frac{T}{R} + \frac{1}{2}G_2\frac{\overline{T}q^2}{R^3} - H\alpha\frac{P}{\overline{q}}, \tag{5.1.20}$$

$$w = H\alpha\Re\left(\frac{T}{q}\right) + H\frac{P}{R}. \tag{5.1.21}$$

Here $\Re$ is the real par sign; H, α, G_1, and G_2 are defined by (9), and

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}. \tag{5.1.22}$$

Formulation of the problem and its solution. Expressions (20) and (21) can be used for the integral equation formulation of the mixed-mixed boundary value problems in an elastic half-space. The boundary conditions in the case of axial symmetry are

$$u = u(\rho), \quad \text{for} \quad 0 \le \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$w = w(\rho), \quad \text{for} \quad 0 \le \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$\sigma = \sigma(\rho), \quad \text{for} \quad a \le \rho \le \infty, \quad 0 \le \phi < 2\pi;$$

$$\tau = \tau(\rho), \quad \text{for} \quad a \le \rho \le \infty, \quad 0 \le \phi < 2\pi. \tag{5.1.23}$$

The set of governing integral equations will take the form

$$2H\left\{-\pi\alpha \int_\rho^a \tau(\rho_0)\mathrm{d}\rho_0 + 2\int_0^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\sigma(\rho_0)\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}}\right\} = \omega_1(\rho), \tag{5.1.24}$$

$$\frac{2H}{\rho}\left\{2\gamma_1\gamma_2 \int_0^\rho \frac{x^2\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\tau(\rho_0)\,\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} - \pi\alpha \int_0^\rho \sigma(\rho_0)\rho_0 \mathrm{d}\rho_0\right\} = \omega_2(\rho). \tag{5.1.25}$$

The functions ω_1 and ω_2 are known from the boundary conditions, and are defined by

$$\omega_1(\rho) = w(\rho) + 2H\left\{\pi\alpha \int_a^\infty \tau(\rho_0)\mathrm{d}\rho_0 - 2\int_a^\infty \frac{\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} \int_a^x \frac{\sigma(\rho_0)\rho_0 \mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}\right\}, \tag{5.1.26}$$

$$\omega_2(\rho) = u(\rho) - 4H\gamma_1\gamma_2\rho \int_a^\infty \frac{\mathrm{d}x}{x^2(x^2 - \rho^2)^{1/2}} \int_a^x \frac{\tau(\rho_0)\,\rho_0^2 \mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}. \tag{5.1.27}$$

The solution to the problem may be presented in the form (Fabrikant, 1986b)

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{f_1(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}, \quad \tau_\rho(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{f_2(t)\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}. \tag{5.1.28}$$

Here σ is the normal traction exerted by the punch, $\tau_\rho = \tau_{\rho z} + i\tau_{\theta z}$, and the stress functions f_1 and f_2 are defined by

$$f(y) = f_1(y) + i\,f_2(y) = \cosh^2(\pi\theta)\left[-\xi(y) + \frac{i}{\pi}\tanh(\pi\theta)\left(\frac{a+y}{a-y}\right)^{i\theta}\int_{-a}^{a}\left(\frac{a+r}{a-r}\right)^{-i\theta}\frac{\xi(r)\mathrm{d}r}{r-y}\right], \tag{5.1.29}$$

where

$$\xi(x) = \frac{1}{\pi^2 H}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\omega_1(\rho)\rho\mathrm{d}\rho}{(x^2-\rho^2)^{1/2}} + \frac{\mathrm{i}}{\sqrt{\gamma_1\gamma_2}}\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\omega_2(\rho)\rho^2\mathrm{d}\rho}{(x^2-\rho^2)^{1/2}}\right\}, \tag{5.1.30}$$

with ω_1 and ω_2 defined by (26) and (27). The general solution simplifies in the case of a bonded punch since $\omega_1 = w$ and $\omega_2 = u$.

Now we need to substitute formulae (28) in (15), modified for the case of distributed loading, and to compute the integrals involved. Here are some details of the derivation. Substitution of the first expression (28) in (15) leads to the integral

$$I_1 = \int_0^{2\pi}\int_0^{a} \ln(R_0 + z)\mathrm{d}\rho_0\mathrm{d}\phi_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a}\frac{f_1(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}. \tag{5.1.31}$$

By interchanging the order of integration in (31), we obtain

$$I_1 = -\int_0^{a} f_1(x)\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^{x}\frac{\ln(R_0+z)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.1.32}$$

The double integral in (32) can be computed by using (A1–A7), with the result

$$I_1 = -2\pi\int_0^{a} f_1(x)\ln\{l_2(x) + [l_2^2(x) - \rho^2]^{1/2}\}\,\mathrm{d}x.$$

The following notation is used throughout this section:

$$l_1(x,\rho,z) \equiv l_1(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} - [(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x,\rho,z) \equiv l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} + [(\rho-x)^2+z^2]^{1/2}\}. \tag{5.1.33}$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notations $l_{1k}(x)$ and $l_{2k}(x)$ are understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for k=1,2.

When substituting the second expression of (28) in (15), we have to remember the relationship between $\tau=\tau_{zx}+i\tau_{yz}$ and $\tau_\rho=\tau_{\rho z}+i\tau_{\theta z}$, namely, $\tau=\tau_\rho e^{i\phi}$. The substitution leads to the integral

$$I_2=\Lambda\int_0^{2\pi}\int_0^a[z\ln(R_0+z)-R_0]e^{-i\phi_0}\rho_0 d\rho_0 d\phi_0\frac{d}{d\rho_0}\int_{\rho_0}^a\frac{f_2(x)\,dx}{(x^2-\rho_0^2)^{1/2}}.$$

Again, interchanging the order of integration, we obtain

$$I_2=-\Lambda\int_0^a f_2(x)\frac{dx}{x}\frac{d}{dx}\int_0^{2\pi}\int_0^x[z\ln(R_0+z)-R_0]e^{-i\phi_0}\frac{\rho_0^2 d\rho_0 d\phi_0}{(x^2-\rho_0^2)^{1/2}}. \qquad (5.1.34)$$

The double integral in (34) can be computed according to (A17–A22), and the final result is rather simple

$$I_2=-2\pi\int_0^a f_2(x)\sin^{-1}\frac{x}{l_2(x)}\,dx. \qquad (5.1.35)$$

Now the potential functions (15) can be expressed through the stress functions as follows:

$$F_1=-\frac{2\pi H}{m_1-1}\left\{\gamma_1\int_0^a f_1(x)\ln[l_{21}(x)+(l_{21}^2(x)-\rho^2)^{1/2}]\,dx+\sqrt{\gamma_1\gamma_2}\int_0^a f_2(x)\sin^{-1}\left(\frac{x}{l_{21}(x)}\right)dx\right\},$$

$$F_2=-\frac{2\pi H}{m_2-1}\left\{\gamma_2\int_0^a f_1(x)\ln[l_{22}(x)+(l_{22}^2(x)-\rho^2)^{1/2}]\,dx+\sqrt{\gamma_1\gamma_2}\int_0^a f_2(x)\sin^{-1}\left(\frac{x}{l_{22}(x)}\right)dx\right\},$$

$$F_3=0. \qquad (5.1.36)$$

These remarkably simple and elementary expressions for the potential functions allow us to obtain the field of displacements and stresses by substitution of (36) in (6) and (12) respectively. The differentiations involved can be performed according to (A27–A45), and the complete solution is

$$u(\rho,\phi,z) = 2\pi H e^{i\phi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{\gamma_k}{\rho}\int_0^a\left[1-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]}\right]f_1(x)\,\mathrm{d}x\right.$$

$$\left.+\sqrt{\gamma_1\gamma_2}\int_0^a\frac{l_{1k}(x)[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}(x)[l_{2k}^2(x)-l_{1k}^2(x)]}f_2(x)\,\mathrm{d}x\right\}, \qquad (5.1.37)$$

$$w(\rho,z) = 2\pi H\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{-\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]}f_1(x)\,\mathrm{d}x\right.$$

$$\left.+\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]}f_2(x)\,\mathrm{d}x\right\}. \qquad (5.1.38)$$

The explicit presence of ϕ in (37) is due to the fact that the notation u stands not for the radial component of the tangential displacement but for a complex representation of its x- and y-components. The field of stresses can be obtained by substitution of (36) in (12), with the result

$$\sigma_1 = 4\pi H A_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[1-\frac{(1+m_k)\gamma_3^2}{\gamma_k^2}\right]\left[z\int_0^a\frac{l_{2k}^4(x)-x^2(2x^2+2z_k^2-\rho^2)}{[l_{2k}^2(x)-x^2]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\right]\right.$$

$$\left.+z_k\sqrt{\gamma_1\gamma_2}\int_0^a\frac{l_{1k}^4(x)-x^2(2x^2+2z_k^2-\rho^2)}{[x^2-l_{1k}^2(x)]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_2(x)\,\mathrm{d}x\right\}, \qquad (5.1.39)$$

$$\sigma_2 = 4\pi H A_{66}\sum_{k=1}^{2}\frac{e^{2i\phi}}{m_k-1}\left\{\gamma_k\int_0^a\left[\frac{2}{\rho^2}\right.\right.$$

$$\left.+\frac{x[x^2-l_1^2(x)]^{1/2}\{\rho^2[6x^2-l_{2k}^2(x)-3l_{1k}^2(x)]-2l_{2k}^4(x)\}}{l_{1k}^2(x)[l_{2k}^2(x)-l_{1k}^2(x)]^3}\right]f_1(x)\,\mathrm{d}x$$

$$\left.-\sqrt{\gamma_1\gamma_2}\int_0^a\frac{x[l_{2k}^2(x)-x^2]^{1/2}\{2l_{1k}^4(x)+\rho^2[l_{1k}^2(x)+3l_{2k}^2(x)-6x^2]\}}{l_{2k}^2(x)[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_2(x)\,\mathrm{d}x\right\},$$

(5.1.40)

$$\sigma_z = \frac{z_k}{\gamma_1 - \gamma_2} \sum_{k=1}^{2} (-1)^{k+1} \left[\gamma_k \int_0^a \frac{l_{2k}^4(x) - x^2(2x^2 + 2z_k^2 - \rho^2)}{[l_2^2(x) - x^2]^{1/2}[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_1(x)\,\mathrm{d}x \right.$$

$$\left. + \sqrt{\gamma_1\gamma_2} \int_0^a \frac{l_{1k}^4(x) - x^2(2x^2 + 2z_k^2 - \rho^2)}{[x^2 - l_{1k}^2(x)]^{1/2}[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_2(x)\,\mathrm{d}x \right], \tag{5.1.41}$$

$$\tau_z = \frac{\rho e^{i\phi}}{\gamma_1 - \gamma_2} \sum_{k=1}^{2} (-1)^{k+1} \left\{ \int_0^a \frac{[l_{2k}^2(x) - x^2]^{1/2}[l_{2k}^2(x) + 3l_{1k}^2(x) - 4x^2]}{[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_1(x)\,\mathrm{d}x \right.$$

$$\left. - \frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \frac{[x^2 - l_{1k}^2(x)]^{1/2}[3l_{2k}^2(x) + l_{1k}^2(x) - 4x^2]}{[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_2(x)\,\mathrm{d}x \right\}. \tag{5.1.42}$$

Formulae (37–42) give the *complete* solution to the bonded punch problem which is the main result of this section. The complete solution for an *isotropic* half-space is readily available as the limiting case (14) of (37–42). Here is the field of displacements:

$$u = \frac{1+\nu}{E} e^{i\phi} \left\{ \int_0^a \left[(1-2\nu)\left(\frac{1}{\rho} - \frac{l_2(x)[x^2 - l_1^2(x)]^{1/2}}{l_1(x)[l_2^2(x) - l_1^2(x)]} \right) \right. \right.$$

$$\left. + \frac{z\rho[l_2^2(x) - x^2]^{1/2}[4x^2 - 3l_1^2(x) - l_2^2(x)]}{[l_2^2(x) - l_1^2(x)]^3} \right] f_1(x)\,\mathrm{d}x$$

$$- \int_0^a \left[2(1-\nu) \frac{l_1(x)[l_2^2(x) - x^2]^{1/2}}{l_2(x)[l_2^2(x) - l_1^2(x)]} \right.$$

$$\left. \left. + \frac{z\rho[x^2 - l_1^2(x)]^{1/2}[4x^2 - l_1^2(x) - 3l_2^2(x)]}{[l_2^2(x) - l_1^2(x)]^3} \right] f_2(x)\,\mathrm{d}x \right\}, \tag{5.1.43}$$

$$w = \frac{1+\nu}{E} \left\{ -\int_0^a \left[2(1-\nu) \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} \right. \right.$$

$$-\frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\Bigg]f_1(x)\,\mathrm{d}x$$

$$+\int_0^a\Bigg[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}-\frac{z^2[l_1^4(x)-x^2(2x^2+2z^2-\rho^2)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\Bigg]f_2(x)\,\mathrm{d}x\Bigg\}. \tag{5.1.44}$$

The limits were computed according to the L'Hôpital rule. The following scheme was used:

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{f(z_1)}{m_1-1}+\frac{f(z_2)}{m_2-1}\right]=-f(z)-\frac{z}{2(1-\nu)}f'(z), \tag{5.1.45}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{m_1-1}+\frac{m_2 f(z_2)}{m_2-1}\right]=f(z)-\frac{z}{2(1-\nu)}f'(z), \tag{5.1.46}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1 f(z_1)}{m_1-1}+\frac{\gamma_2 f(z_2)}{m_2-1}\right]=-\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \tag{5.1.47}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{\gamma_1(m_1-1)}+\frac{m_2 f(z_2)}{\gamma_2(m_2-1)}\right]=\frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)}. \tag{5.1.48}$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1=1,\quad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial\gamma_1}\right]=2(1-\nu), \tag{5.1.49}$$

and the symbol (´) indicates differentiation with respect to z. The derivation of the field of stresses for the case of isotropy is left to the reader.

Applications. Consider the case of a flat circular centrally loaded punch bonded to a transversely isotropic elastic half-space $z\geq 0$. The stress functions in this case are (Fabrikant, 1986b)

$$f_1(x)=-\frac{w_0}{\pi^2 H}\cosh\pi\theta\, Y_c(x),\qquad f_2(x)=-\frac{w_0}{\pi^2 H}\cosh\pi\theta\, Y_s(x),$$

$$Y_{c}(x)=\cos\left(\theta\ln\frac{a+x}{a-x}\right),\qquad Y_{s}(x)=\sin\left(\theta\ln\frac{a+x}{a-x}\right),\tag{5.1.50}$$

where w_0 is the punch settlement, and

$$\theta=\frac{1}{2\pi}\ln\frac{\sqrt{\gamma_1\gamma_2}+\alpha}{\sqrt{\gamma_1\gamma_2}-\alpha}.$$

The complete solution can be obtained by substitution of (50) in (37–42). In the case of isotropy, the value of θ is defined by $\theta=(1/2\pi)\ln(3-4\nu)$. We have performed some computations in order to compare the field of displacements around a flat bonded punch with similar results for a smooth punch. The importance of such a comparison lies in the fact that the smooth and bonded punches represent two extreme cases of interaction between a punch and an elastic half-space, and usually give upper and lower bounds for various parameters of practical interest. The computations were made for the case of isotropy. The field of displacements around a bonded punch is

$$u=\frac{w_0 e^{i\phi}\cosh\pi\theta}{\pi(1-\nu)}\left\{-\int_0^a\left[(1-2\nu)\left(\frac{1}{\rho}-\frac{l_2(x)[x^2-l_1^2(x)]^{1/2}}{l_1(x)[l_2^2(x)-l_1^2(x)]}\right)\right.\right.$$

$$\left.+\frac{z\rho[l_2^2(x)-x^2]^{1/2}[4x^2-3l_1^2(x)-l_2^2(x)]}{[l_2^2(x)-l_1^2(x)]^3}\right]Y_c(x)\,\mathrm{d}x$$

$$+\int_0^a\left[2(1-\nu)\frac{l_1(x)[l_2^2(x)-x^2]^{1/2}}{l_2(x)[l_2^2(x)-l_1^2(x)]}\right.$$

$$\left.\left.+\frac{z\rho[x^2-l_1^2(x)]^{1/2}[4x^2-l_1^2(x)-3l_2^2(x)]}{[l_2^2(x)-l_1^2(x)]^3}\right]Y_s(x)\,\mathrm{d}x\right\},\tag{5.1.51}$$

$$w=\frac{w_0\cosh\pi\theta}{\pi(1-\nu)}\left\{\int_0^a\left[2(1-\nu)\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}-\frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right]Y_c(x)\,\mathrm{d}x\right.$$

$$\left.-\int_0^a\left[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}-\frac{z^2[l_1^4(x)-x^2(2x^2+2z^2-\rho^2)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right]Y_s(x)\,\mathrm{d}x\right\}.$$

(5.1.52)

The field of displacements in the case of a smooth punch takes the form (Fabrikant, 1989b)

$$u = \frac{w_0 e^{i\phi}}{\pi(1-\nu)} \left\{ -(1-2\nu)\left[\frac{a-(a^2-l_1^2)^{1/2}}{\rho}\right] + \frac{z l_1 (l_2^2-a^2)^{1/2}}{l_2(l_2^2-l_1^2)} \right\}, \tag{5.1.53}$$

$$w = \frac{2w_0}{\pi}\left[\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{z(a^2-l_1^2)^{1/2}}{2(1-\nu)(l_2^2-l_1^2)}\right]. \tag{5.1.54}$$

The solutions (51–52) and (53–54) depend essentially on one parameter, namely, Poisson coefficient. In the case $\nu = 1/2$, both solutions coincide. It would be a good exercise for the reader to prove this by direct integration of (51–52) which should yield (53–54). The greatest difference between solutions is attained for the Poisson coefficient $\nu = 0$. This value was taken in numerical computations. The results are shown in Fig. 5.1.1 (the ratio u/w_0 versus ρ/a) and Fig. 5.1.2 (the ratio w/w_0 versus ρ/a) for $z/a = 0, 0.1, 0.5, 1.0$. We took $a = 1$ in all computations. The solid line curves correspond to the case of a bonded punch,

Fig. 5.1.1. The field of radial displacements

while the broken line curves give similar results for a smooth punch. As we could expect, the field of radial displacements under the bonded punch differs quite significantly from that of a smooth punch. On the contrary, the field of normal displacements differs very little from the case of a smooth punch, with

Fig. 5.1.2. The field of normal displacements

the maximum deviation not exceeding $0.06w_0$ at $z=0$, $\rho=1.1a$.

The complete solution (37–42) may be used for solving more complicated problems of interactions between bonded punches and anchor loads.

5.2. Inclined bonded circular punch

A complete solution is given to the problem of an inclined circular punch bonded to a transversely isotropic elastic half-space. Explicit expressions are derived for the field of stresses and displacements around such punch subjected to a shifting load and a tilting moment. The complete solution is initially expressed in terms of three potential functions. The displacements are defined through first derivatives of the potential functions, while the stresses are given by second derivatives. The complete solution for an isotropic body is obtained as a limiting case of the general solution. Specific computations are performed in order to compare the elastic field in the vicinity of a bonded punch with similar parameters for a smooth punch. Its is found that the influence of bonding on normal displacement is relatively small, while its influence on tangential displacements may be quite significant.

The problem of a bonded circular punch subjected to a shifting force and a tilting moment was first considered in (Fabrikant, 1971c). All known solutions define the elastic field in the plane z=0 only. We give below a complete solution.

The boundary conditions in the case of a flat circular bonded punch subjected to a shifting force and a tilting moment are

$$u = u_0 = \text{const.}, \qquad \text{for } 0\le\rho\le a, \qquad 0\le\phi<2\pi;$$

$$w = -\delta\rho\cos\phi, \qquad \text{for } 0\le\rho\le a, \qquad 0\le\phi<2\pi;$$

$$\sigma = 0, \qquad \text{for } a\le\rho\le\infty, \qquad 0\le\phi<2\pi;$$

$$\tau = 0, \qquad \text{for } a\le\rho\le\infty, \qquad 0\le\phi<2\pi. \tag{5.2.1}$$

The set of governing integral equations will take the form (Fabrikant, 1971c)

$$\frac{2G_1}{\rho^2}\int_0^\rho \frac{x^4\,dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\tau_2(\rho_0)d\rho_0}{\rho_0(\rho_0^2-x^2)^{1/2}} - \frac{2\pi H\alpha}{\rho^2}\int_0^\rho \sigma_1(\rho_0)\rho_0^2 d\rho_0$$

$$+\frac{2G_2}{\rho^2}\int_0^\rho \frac{\rho^2-2x^2}{(\rho^2-x^2)^{1/2}}dx\int_x^a \frac{\bar{\tau}_0(\rho_0)\rho_0 d\rho_0}{(\rho_0^2-x^2)^{1/2}} = 0, \tag{5.2.2}$$

$$2G_2\int_0^\rho \frac{dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{(\rho_0^2-2x^2)\,\bar{\tau}_2(\rho_0)d\rho_0}{\rho_0(\rho_0^2-x^2)^{1/2}} + 2\pi H\alpha\int_\rho^a \sigma_{-1}(\rho_0)d\rho_0$$

$$+2G_1\int_0^\rho \frac{dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\tau_0(\rho_0)\rho_0 d\rho_0}{(\rho_0^2-x^2)^{1/2}} = u_0, \tag{5.2.3}$$

$$2\pi H\alpha\,\Re\left\{\frac{e^{-i\phi}}{\rho}\int_0^\rho \tau_0(\rho_0)\rho_0 d\rho_0 - \rho e^{i\phi}\int_\rho^a \tau_2(\rho_0)\frac{d\rho_0}{\rho_0}\right\}$$

$$+\frac{4H}{\rho}\int_0^\rho \frac{x^2\,dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\sigma_1(\rho_0)e^{i\phi}+\sigma_{-1}(\rho_0)e^{-i\phi}}{(\rho_0^2-x^2)^{1/2}}d\rho_0 = -\delta\rho\cos\phi. \tag{5.2.4}$$

The structure of equations (2–4) is such that we may assume all the unknown functions τ_0, τ_2, σ_1, and σ_{-1} to be real. The solution may be represented in the form

$$\sigma_1(\rho) = \sigma_{-1}(\rho) = \frac{d}{d\rho}\int_0^{\rho} \frac{f(t)dt}{(\rho^2 - t^2)^{1/2}},$$

$$\tau_0(\rho) = \bar{\tau}_0(\rho) = -\frac{C}{\rho}\frac{d}{d\rho}\int_{\rho}^{a} \frac{f(t)t\,dt}{(t^2 - \rho^2)^{1/2}} + \frac{D}{(a^2 - \rho^2)^{1/2}},$$

$$\tau_2(\rho) = \bar{\tau}_2(\rho) = -C\rho\frac{d}{d\rho}\left\{\frac{1}{\rho^2}\int_{\rho}^{a} \frac{f(t)t\,dt}{(t^2 - \rho^2)^{1/2}}\right\} - D\frac{2a^2 - \rho^2}{\rho^2(a^2 - \rho^2)^{1/2}}. \tag{5.2.5}$$

Here C and D are the constants to be determined, and f is the stress function. They are defined as follows (Fabrikant, 1989a):

$$f(t) = -\frac{\delta\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}\left[tY_s(t) - \theta a Y_c(t)\right] + AY_c(t). \tag{5.2.6}$$

$$C = \frac{\alpha}{\gamma_1\gamma_2}, \quad D = \frac{\pi\theta\alpha}{\gamma_1\gamma_2\sinh(\pi\theta)}A,$$

$$Y_{c,s}(t) = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left[\theta\ln\left(\frac{a+t}{a-t}\right)\right], \quad \tanh(\pi\theta) = \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \tag{5.2.7}$$

$$A = \left(u_0 + \frac{\delta a\theta\alpha}{\tanh(\pi\theta)}\right)\left[\frac{\pi^2 H\alpha}{\cosh(\pi\theta)}\left(1 + \frac{\pi\theta(G_1 + G_2)}{\tanh(\pi\theta)(G_1 - G_2)}\right)\right]^{-1}. \tag{5.2.8}$$

The displacements of the punch are related to the applied loading as

$$u_0 = \frac{1}{8a}\left[\pi(G_1 + G_2) + \frac{(1 + 4\theta^2)\tanh(\pi\theta)}{\theta(1 + \theta^2)}(G_1 - G_2)\right]T - \frac{3H\alpha}{4a^2(1 + \theta^2)}M,$$

$$\delta = \frac{3H\alpha}{4a^2(1 + \theta^2)}\left[-T + \frac{M}{a\theta\sqrt{\gamma_1\gamma_2}}\right]. \tag{5.2.9}$$

In order to proceed further, we need to express the normal stress distribution in a form slightly different from the first expression (5), namely,

$$\sigma_1(\rho) = \sigma_{-1}(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{f_1(t)\mathrm{d}t}{(t^2-\rho^2)^{1/2}}. \tag{5.2.10}$$

Here f_1 is a new stress function which can be related to f by an easily verifiable expression

$$f_1(x) = -\frac{2}{\pi}x\int_0^a \frac{(a^2-t^2)^{1/2}f(t)\mathrm{d}t}{(a^2-x^2)^{1/2}(t^2-x^2)}. \tag{5.2.11}$$

Substitution of (6) in (11) yields, after integration (see Appendix A3.1 in Fabrikant, 1989a)

$$f_1(t) = \frac{\delta\cosh(\pi\theta)}{\pi^2 H}\left[tY_\mathrm{c}(t) + \theta a Y_\mathrm{s}(t)\right] + \tanh(\pi\theta)AY_\mathrm{s}(t). \tag{5.2.12}$$

We have dropped here the term const$\cdot t/(a^2 - t^2)^{1/2}$ since its addition to or subtraction from f_1 does not change the value of normal stress defined by (10).

Now we need to substitute the last two formulae (5) and (10) in (5.1.15), modified for the case of distributed loading, and to compute the integrals involved. Here are some details of the derivation. Substitution of expression (10) in (5.1.15) leads to the integral

$$I_1 = \int_0^{2\pi}\int_0^a \ln(R_0+z)\mathrm{e}^{i\phi_0}\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0\,\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a \frac{f_1(x)\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.13}$$

By interchanging the order of integration in (13), we obtain

$$I_1 = -\int_0^a \frac{1}{x}f_1(x)\,\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x \frac{\mathrm{e}^{i\phi_0}\ln(R_0+z)\rho_0^2\mathrm{d}\rho_0\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.14}$$

The double integral in (14) can be computed by using (A17–A20), with the result

$$I_1 = 2\pi\frac{\mathrm{e}^{i\phi}}{\rho}\int_0^a \{x-[x^2-l_1^2(x)]^{1/2}\}f_1(x)\,\mathrm{d}x. \tag{5.2.15}$$

The following notation is used throughout this section:

$$l_1(x,\rho,z) \equiv l_1(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} - [(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x,\rho,z) \equiv l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} + [(\rho-x)^2+z^2]^{1/2}\}. \tag{5.2.16}$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notations $l_{1k}(x)$ and $l_{2k}(x)$ are understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for k=1,2,3.

Substitution of the second expression of (5) in (5.1.15) leads to the integral

$$I_2 = -\Lambda\int_0^{2\pi}\int_0^a [z\ln(R_0+z)-R_0]\mathrm{d}\rho_0\mathrm{d}\phi_0\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a \frac{f(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.17}$$

Again, interchanging the order of integration, we obtain

$$I_2 = \Lambda\int_0^a f(x)\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x [z\ln(R_0+z)-R_0]\frac{\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.18}$$

The double integral in (18) can be computed according to (A1–A7), and the final result is rather simple

$$I_2 = 2\pi\frac{e^{i\phi}}{\rho}\int_0^a \{z-[l_2^2(x)-x^2]^{1/2}\}f(x)\,\mathrm{d}x. \tag{5.2.19}$$

Substitution of the third expression of (5) in (5.1.15) yields the integral

$$I_3 = \overline{\Lambda}\int_0^{2\pi}\int_0^a [z\ln(R_0+z)-R_0]\rho_0^2 e^{2i\phi_0}\,\mathrm{d}\rho_0\mathrm{d}\phi_0\frac{\mathrm{d}}{\mathrm{d}\rho_0}\left\{\frac{1}{\rho_0^2}\int_{\rho_0}^a \frac{f(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}\right\}.$$

Interchanging the order of integration, we obtain

$$I_3 = \overline{\Lambda}\int_0^a f(x)\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x e^{2i\phi_0}[z\ln(R_0+z)-R_0]\frac{(\rho_0^2-2x^2)\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho_0(x^2-\rho_0^2)^{1/2}}. \tag{5.2.20}$$

Again, the double integral in (20) can be computed according to (A8–A14), and the final result is

$$I_3 = 2\pi\frac{e^{i\phi}}{\rho}\int_0^a \{[l_2^2(x) - x^2]^{1/2} - (\rho^2 + z^2)^{1/2}\} f(x)\,dx. \tag{5.2.21}$$

Now the potential functions can be expressed through the stress functions as follows:

$$F_1 = \frac{4\pi H}{m_1 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_1 - [l_{21}^2(x) - x^2]^{1/2}\} f(x)\,dx\right.$$

$$\left. + \gamma_1\int_0^a \{x - [x^2 - l_{11}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_2 = \frac{4\pi H}{m_2 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_2 - [l_{22}^2(x) - x^2]^{1/2}\} f(x)\,dx\right.$$

$$\left. + \gamma_2\int_0^a \{x - [x^2 - l_{12}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_3 = D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\int_0^a \{z_3 - [l_{23}^2(x) - x^2]^{1/2}\}\,dx$$

$$= D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\left[z_3 a - \frac{(l_{23}^2 - a^2)^{1/2}(2a^2 - l_{13}^2)}{2a} - \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_{23}}\right)\right] \tag{5.2.22}$$

These remarkably simple and elementary expressions for the potential functions allow us to obtain the field of displacements and stresses by substitution of (22) in (5.1.6) and (5.1.12) respectively. The differentiations involved can be performed according to (A27–A45), and the complete solution is

$$u(\rho,\phi,z) = 4\pi H\sum_{k=1}^{2}\frac{1}{m_k - 1}\left\{-\frac{e^{2i\phi}}{\rho^2}\left[\alpha\int_0^a \{z_k - [l_{2k}^2(x) - x^2]^{1/2}\} f(x)\,dx\right.\right.$$

$$+\gamma_k\int_0^a \{x-[x^2-l_{1k}^2(x)]^{1/2}\}f_1(x)\,\mathrm{d}x\Bigg]$$

$$+\frac{\mathrm{e}^{2i\phi}+1}{2}\left[-\alpha\int_0^a \frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f(x)\,\mathrm{d}x+\gamma_k\int_0^a \frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\right]\Bigg\}$$

$$-D\frac{2\gamma_3}{A_{44}}\left\{\frac{\mathrm{e}^{2i\phi}}{\rho^2}\left[z_3 a-(l_{23}^2-a^2)^{1/2}\left(a-\frac{l_{13}^2}{2a}\right)\right]-\frac{1}{2}\sin^{-1}\left(\frac{a}{l_{23}}\right)\right\}, \tag{5.2.23}$$

$$w(\rho,z)=4\pi H\frac{\cos\phi}{\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\alpha}{\gamma_k}\int_0^a\left(1-\frac{l_{2\mathrm{k}}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)f(x)\,\mathrm{d}x\right.\right.$$

$$\left.\left.-\int_0^a \frac{l_{1\mathrm{k}}(x)[\rho^2-l_{1\mathrm{k}}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\right]\right\}. \tag{5.2.24}$$

We recall that the stress functions f and f_1 are defined by (6–8) and (11) respectively. The field of stresses can be obtained by substitution of (22) in (5.1.12), with the result

$$\sigma_1=8\pi HA_{66}\rho\cos\phi\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[1\right.\right.$$

$$-\frac{(1+m_k)\gamma_3^2}{\gamma_k^2}\Bigg]\left[\alpha\int_0^a \frac{[l_{2k}^2(x)-x^2]^{1/2}[4x^2-3l_{1k}^2(x)-l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x\right.$$

$$\left.\left.-\gamma_k\int_0^a \frac{[x^2-l_{1k}^2(x)]^{1/2}[4x^2-l_{1k}^2(x)-3l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\right]\right\}, \tag{5.2.25}$$

$$\sigma_2=8\pi HA_{66}\sum_{k=1}^{2}\frac{\mathrm{e}^{i\phi}}{m_k-1}\left\{\frac{4\mathrm{e}^{2i\phi}}{\rho^3}\left[\alpha\int_0^a\left(z_k-[l_{2k}^2(x)-x^2]^{1/2}+\frac{\rho^2[l_{2k}^2(x)-x^2]^{1/2}}{2[l_{2k}^2(x)-l_{1k}^2(x)]}\right)f(x)\,\mathrm{d}x\right.\right.$$

$$+\gamma_k\int_0^a\left(x-[x^2-l_{1k}^2(x)]^{1/2}-\frac{\rho^2[x^2-l_{1k}^2(x)]^{1/2}}{2[l_{2k}^2(x)-l_{1k}^2(x)]}\right)f_1(x)\,\mathrm{d}x\Bigg]$$

$$+\frac{\mathrm{e}^{2i\phi}+1}{2}\rho\Bigg[-\alpha\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}[4x^2-3l_{1k}^2(x)-l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x$$

$$+\gamma_k\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}[4x^2-l_{1k}^2(x)-3l_{2k}^2(x)]f_1(x)\,\mathrm{d}x\Bigg]\Bigg\}$$

$$+4D\frac{\mathrm{e}^{i\phi}}{\gamma_3}\left\{\frac{4\mathrm{e}^{2i\phi}}{\rho^3}\left[z_3a-(l_{23}^2-a^2)^{1/2}\left(a-\frac{l_{13}^2}{2a}\right)\right]-\frac{1-\mathrm{e}^{2i\phi}}{2}\frac{l_{13}(l_{23}^2-a^2)^{1/2}}{l_{23}(l_{23}^2-l_{13}^2)}\right\}. \quad (5.2.26)$$

$$\sigma_z=\frac{2\rho\cos\phi}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{k+1}\Bigg\{\alpha\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}[4x^2-3l_{1k}^2(x)-l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x$$

$$-\gamma_k\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}[4x^2-l_{1k}^2(x)-3l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\Bigg\}, \quad (5.2.27)$$

$$\tau_z=\frac{2}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{\mathrm{k}}\Bigg\{-\frac{\mathrm{e}^{2i\phi}}{\rho^2}\Bigg[\frac{\alpha}{\gamma_k}\int_0^a\frac{l_{2\mathrm{k}}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f(x)\,\mathrm{d}x$$

$$+\int_0^a\frac{l_{1\mathrm{k}}(x)[\rho^2-l_{1\mathrm{k}}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\Bigg]$$

$$+\frac{\mathrm{e}^{2i\phi}+1}{2}\Bigg[\frac{\alpha z_k}{\gamma_k}\int_0^a\frac{x^2(2x^2+2z_k^2-\rho^2)-l_{2k}^4(x)}{[l_{2k}^2(x)-x^2]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x$$

$$-z_k\int_0^a\frac{l_{2k}^4(x)-x^2(2x^2+2z_k^2-\rho^2)}{[x^2-l_{1k}^2(x)]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\Bigg]\Bigg\}$$

$$+ 2D(a^2 - l_{13}^2)^{1/2}\left[\frac{\mathrm{e}^{2i\phi}}{\rho^2} - \frac{1-\mathrm{e}^{2i\phi}}{2(l_{23}^2 - l_{13}^2)}\right]. \tag{5.2.28}$$

Formulae (23–28) give the *complete* solution to the bonded punch problem which is the main result of this section. The complete solution for an *isotropic* half-space is readily available as the limiting case (5.1.14) of (23–28). Here is the field of displacements:

$$u = \frac{1}{\mu}\left\{\frac{\mathrm{e}^{2i\phi}}{\rho^2}(1-2\nu)\int_0^a\left[z - [l_2^2(x) - x^2]^{1/2} + \frac{z}{2(1-\nu)}\left(1 - \frac{l_2(x)[l_2^2(x)-\rho^2]^{1/2}}{l_2^2(x) - l_1^2(x)}\right)\right]f(x)\,\mathrm{d}x\right.$$

$$+ \frac{\mathrm{e}^{2i\phi}}{\rho^2}\int_0^a\left[(1-2\nu)[x - [x^2 - l_1^2(x)]^{1/2}] - \frac{z\,l_1(x)[\rho^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\right]f_1(x)\,\mathrm{d}x$$

$$+ \frac{\mathrm{e}^{2i\phi}+1}{2}(1-2\nu)\int_0^a\left[\frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} + \frac{z^2[x^2(2x^2+2z^2-\rho^2) - l_2^4(x)]}{[l_2^2(x) - x^2]^{1/2}[l_2^2(x) - l_1^2(x)]^3}\right]f(x)\,\mathrm{d}x$$

$$\left. - \frac{\mathrm{e}^{2i\phi}+1}{2}\int_0^a\left[(1-2\nu)\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)} + \frac{z^2[l_1^4(x) - x^2(2x^2+2z^2-\rho^2)]}{[x^2 - l_1^2(x)]^{1/2}[l_2^2(x) - l_1^2(x)]^3}\right]f_1(x)\,\mathrm{d}x\right\}$$

$$- \frac{2D}{\mu}\left\{\frac{\mathrm{e}^{2i\phi}}{\rho^2}\left[za - (l_2^2 - a^2)^{1/2}\left(a - \frac{l_1^2}{2a}\right)\right] - \frac{1}{2}\sin^{-1}\left(\frac{a}{l_2}\right)\right\}, \tag{5.2.29}$$

$$w = \frac{\cos\phi}{\mu\rho}\left\{\frac{(1-2\nu)}{2(1-\nu)}\int_0^a\left[(1-2\nu)\left(1 - \frac{l_2(x)[l_2^2(x) - \rho^2]^{1/2}}{l_2^2(x) - l_1^2(x)}\right) - \frac{z\rho^2[l_2^2(x) - x^2]^{1/2}}{[l_2^2(x) - l_1^2(x)]^3}[4x^2\right.\right.$$

$$\left. - 3l_1^2(x) - l_2^2(x)]\right]f(x)\,\mathrm{d}x - \int_0^a\left[2(1-\nu)\frac{l_1(x)[\rho^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\right.$$

$$\left.\left. - \frac{z\rho^2[x^2 - l_1^2(x)]^{1/2}}{[l_2^2(x) - l_1^2(x)]^3}[4x^2 - l_1^2(x) - 3l_2^2(x)]\right]f_1(x)\,\mathrm{d}x\right\}. \tag{5.2.30}$$

Here μ stands for the shear modulus, and ν is the Poisson coefficient. In the case of isotropy, the value of θ is defined by θ=(1/2π)ln(3−4ν). Formulae (6–8) and (12) in the case of isotropy take the form

$$f(x)=\frac{4}{\pi}\mu\cosh(\pi\theta)\left\{\frac{\delta}{(1-2\nu)}\left[-xY_{\mathrm{s}}(x)+\theta aY_{\mathrm{c}}(x)\right]+\frac{u_0+\delta a\theta}{1-2\nu+2\pi\theta}Y_{\mathrm{c}}(x)\right\},$$

$$f_1(x)=\frac{4}{\pi}\mu\sinh(\pi\theta)\left\{\frac{\delta}{(1-2\nu)}\left[xY_{\mathrm{c}}(x)+\theta aY_{\mathrm{s}}(x)\right]+\frac{u_0+\delta a\theta}{1-2\nu+2\pi\theta}Y_{\mathrm{s}}(x)\right\},$$

$$A=\frac{4\mu\cosh(\pi\theta)(u_0+\delta a\theta)}{\pi(1-2\nu+2\pi\theta)},\quad D=\frac{4\mu\theta(u_0+\delta a\theta)}{1-2\nu+2\pi\theta}$$

The limits were computed according to the L'Hôpital rule, and the symbol (´) indicates differentiation with respect to z. The derivation of the field of stresses for the case of isotropy is left to the reader.

We have performed some computations in order to compare the field of displacements around a flat inclined bonded punch with similar results for a smooth punch. The importance of such a comparison lies in the fact that the smooth and bonded punches represent two extreme cases of interaction between a punch and an elastic half-space, and usually give upper and lower bounds for various parameters of practical interest. The field of displacements in the case of a smooth punch takes the form (Fabrikant, 1989a)

$$u=-\frac{2\delta\rho\cos\phi}{\pi}\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\left\{z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right.$$
$$\left.+\,\mathrm{e}^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right\},\qquad(5.2.31)$$

$$w=-\frac{2}{\pi}\delta\rho\cos\phi\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left[\sin^{-1}\left(\frac{a}{l_{2k}}\right)-\frac{a(l_{2k}^2-a^2)^{1/2}}{l_{2k}^2}\right].\qquad(5.2.32)$$

The solutions (23–24) and (31–32) in the case of θ=0 coincide. It would be a good exercise for the reader to prove this by a direct integration of (23–24) which should yield (31–32). In the case of isotropy, this situation corresponds to the value of Poisson coefficient ν=1/2.

The computations were made for the case of isotropy, with the following numerical values assigned to the parameters: $u_0=0$, $\delta=1$, and $a=1.3$. Formula (32) in this case takes the form

$$w=-\frac{2}{\pi}\delta\rho\cos\phi\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{a}{l_2^2}(l_2^2-a^2)^{1/2}+\frac{za^2(a^2-l_1^2)^{1/2}}{(1-\nu)l_2^2(l_2^2-l_1^2)}\right]. \qquad (5.2.33)$$

The greatest difference between solutions (30) and (33) is attained for the Poisson coefficient ν=0. This value was taken in numerical computations. The results for $\delta=1$, $a=1.3$ (w versus ρ/a) are presented in Fig. 5.2.1. As before, the solid

Fig. 5.2.1. The influence of bonding on normal displacements

line curves correspond to the case of a bonded punch, while the broken line curves give the relevant results for a smooth punch. It was found that the field of normal displacements due to a bonded punch differs very little from the case of a smooth punch, with the maximum deviation not exceeding 3.5% at z=0, $\rho=1.1a$. On the contrary, it is evident, that the field of tangential displacements of a bonded punch might differ quite significantly from that of a smooth punch.

5.3. Interaction of a normal load with a bonded punch

We consider a transversely isotropic elastic half-space $z \geq 0$. A flat circular punch of radius a is bonded to its boundary $z=0$, with the punch centre coinciding with the coordinate system origin $\rho=0$. Let a point force N (Fig. 5.3.1) be applied in the Oz direction at the point with the polar cylindrical coordinates (ρ,ϕ,z). We may assume, without loss of generality, that $\phi=0$. We

Fig. 5.3.1. Geometry of the problem

need to find the punch settlement w_N, its tangential displacement u_N, and the angle of inclination δ_N which are due to the point load N. The reader is reminded that the punch settlement is understood as the normal displacement of the punch centre; the angle of inclination is the angle between the punch base and the plane $z=0$.

First of all, we need to solve two auxiliary problems, namely, the one of a centrally loaded bonded punch, and the second one is the problem of an inclined bonded punch. We consider below each problem separately, after which, the reciprocal theorem is used to obtain the solution to the main problem.

Problem 1. We consider the mixed-mixed problem characterized by the following boundary conditions:

$$u = 0, \qquad \text{for } 0 \leq \rho \leq a, \qquad 0 \leq \phi < 2\pi;$$

$$w = w_0, \qquad \text{for } 0 \leq \rho \leq a, \qquad 0 \leq \phi < 2\pi;$$

$$\sigma = 0, \qquad \text{for } a \leq \rho \leq \infty, \qquad 0 \leq \phi < 2\pi;$$

$$\tau = 0, \qquad \text{for } a \leq \rho \leq \infty, \qquad 0 \leq \phi < 2\pi. \tag{5.3.1}$$

The solution to the problem may be presented in the form (Fabrikant, 1986b)

$$\sigma(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\int_\rho^a \frac{f_1(t)t\,dt}{(t^2-\rho^2)^{1/2}}, \quad \tau_\rho(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{d}{d\rho}\int_\rho^a \frac{f_2(t)\,dt}{(t^2-\rho^2)^{1/2}}. \tag{5.3.2}$$

Here σ is the normal traction exerted by the punch, $\tau_\rho = \tau_{\rho z} + i\tau_{\theta z}$, and the stress functions f_1 and f_2 are defined in this particular case by

$$f_1(x) = -\frac{w_0}{\pi^2 H}\cosh\pi\theta \cos\left(\theta \ln\frac{a+x}{a-x}\right),$$

$$f_2(x) = -\frac{w_0}{\pi^2 H}\cosh\pi\theta \sin\left(\theta \ln\frac{a+x}{a-x}\right),$$

$$\theta = \frac{1}{2\pi}\ln\frac{\sqrt{\gamma_1\gamma_2}+\alpha}{\sqrt{\gamma_1\gamma_2}-\alpha}. \tag{5.3.3}$$

where w_0 is the punch settlement. In the case of isotropy, the value of θ is defined by $\theta=(1/2\pi)\ln(3-4\nu)$. Repeating the transformations from section 5.1, leading to (5.1.36), we come to the following expressions for the potential functions

$$F_1 = \frac{2w_0\cosh(\pi\theta)}{\pi(m_1-1)}\left\{\gamma_1\int_0^a Y_c(x)\ln[l_{21}(x)+(l_{21}^2(x)-\rho^2)^{1/2}]\,dx\right.$$

$$\left. + \sqrt{\gamma_1\gamma_2}\int_0^a Y_s(x)\sin^{-1}\left(\frac{x}{l_{21}(x)}\right)dx\right\},$$

$$F_2 = \frac{2w_0\cosh(\pi\theta)}{\pi(m_2-1)}\left\{\gamma_2\int_0^a Y_c(x)\ln[l_{22}(x)+(l_{22}^2(x)-\rho^2)^{1/2}]\,dx\right.$$

$$\left. + \sqrt{\gamma_1\gamma_2}\int_0^a Y_s(x)\sin^{-1}\left(\frac{x}{l_{22}(x)}\right)dx\right\},$$

$$F_3 = 0. \tag{5.3.4}$$

where w_0 is the punch settlement, θ is defined by (3), and

$$Y_c(x) = \cos\left(\theta \ln\frac{a+x}{a-x}\right), \quad Y_s(x) = \sin\left(\theta \ln\frac{a+x}{a-x}\right). \tag{5.3.5}$$

We need only the expression for the normal displacement

$$w(\rho,z) = \frac{2}{\pi}w_0\cosh(\pi\theta)\sum_{k=1}^{2} \frac{m_k}{m_k-1}\left\{\int_0^a \frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\,Y_c(x)\,\mathrm{d}x\right.$$

$$\left. -\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a \frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\,Y_s(x)\,\mathrm{d}x\right\}. \tag{5.3.6}$$

Taking into consideration the relationship between the punch settlement w_0 and the applied to the punch force P (Fabrikant, 1989a)

$$w_0 = \frac{PH\tanh(\pi\theta)}{2a\theta}, \tag{5.3.7}$$

expression (6) can be rewritten as

$$w(\rho,z) = \frac{\mathrm{PH}\sinh(\pi\theta)}{\pi a\theta}\sum_{k=1}^{2} \frac{m_k}{m_k-1}\left\{\int_0^a \frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\,Y_c(x)\,\mathrm{d}x\right.$$

$$\left. -\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a \frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\,Y_s(x)\,\mathrm{d}x\right\}. \tag{5.3.8}$$

Problem 2. The boundary conditions in the case of a flat circular bonded punch subjected to a shifting force T and a tilting moment M are

$u = u_0 = \text{const.},$ for $0 \le \rho \le a,$ $0 \le \phi < 2\pi;$

$w = -\delta\rho\cos\phi,$ for $0 \le \rho \le a,$ $0 \le \phi < 2\pi;$

$\sigma = 0,$ for $a \le \rho \le \infty,$ $0 \le \phi < 2\pi;$

$$\tau = 0, \qquad \text{for } a \le \rho \le \infty, \qquad 0 \le \phi < 2\pi. \tag{5.3.9}$$

Here u_0 is the tangential displacement of the punch, and δ is the angle of inclination. Again, from the results in section 5.2 we get the potential functions

$$F_1 = \frac{4\pi H}{m_1 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_1 - [l_{21}^2(x) - x^2]^{1/2}\} f(x)\,\mathrm{d}x\right.$$

$$\left. + \gamma_1\int_0^a \{x - [x^2 - l_{11}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_2 = \frac{4\pi H}{m_2 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_2 - [l_{22}^2(x) - x^2]^{1/2}\} f(x)\,\mathrm{d}x\right.$$

$$\left. + \gamma_2\int_0^a \{x - [x^2 - l_{12}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_3 = D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\int_0^a \{z_3 - [l_{23}^2(x) - x^2]^{1/2}\}\,\mathrm{d}x$$

$$= D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\left[z_3 a - \frac{(l_{23}^2 - a^2)^{1/2}(2a^2 - l_{13}^2)}{2a} - \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_{23}}\right)\right] \tag{5.3.10}$$

Here f and f_1 are the stress functions and D is a constant. They are defined (Fabrikant, 1989a) as follows:

$$f(t) = -\frac{\delta\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}\left[tY_{\mathrm{s}}(t) - \theta a Y_{\mathrm{c}}(t)\right] + AY_{\mathrm{c}}(t). \tag{5.3.11}$$

$$f_1(t) = \frac{\delta\cosh(\pi\theta)}{\pi^2 H}\left[tY_{\mathrm{c}}(t) + \theta a Y_{\mathrm{s}}(t)\right] + \tanh(\pi\theta)AY_{\mathrm{s}}(t). \tag{5.3.12}$$

$$D = \frac{\pi\theta\alpha}{\gamma_1\gamma_2\sinh(\pi\theta)}A, \qquad \tanh(\pi\theta) = \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \tag{5.3.13}$$

$$A = \left(u_0 + \frac{\delta a\theta\alpha}{\tanh(\pi\theta)}\right)\left[\frac{\pi^2 H\alpha}{\cosh(\pi\theta)}\left(1 + \frac{\pi\theta(G_1+G_2)}{\tanh(\pi\theta)(G_1-G_2)}\right)\right]^{-1}. \qquad (5.3.14)$$

Again, we need only the expression for the normal displacement

$$w(\rho,z) = 4\pi H\frac{\cos\phi}{\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\alpha}{\gamma_k}\int_0^a\left(1-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)f(x)\,\mathrm{d}x\right.\right.$$

$$\left.\left.-\int_0^a\frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\right]\right\}. \qquad (5.3.15)$$

The displacements of the punch are related to the applied loading as (Fabrikant, 1989a)

$$u_0 = \frac{1}{8a}\left[\pi(G_1+G_2) + \frac{(1+4\theta^2)\tanh(\pi\theta)}{\theta(1+\theta^2)}(G_1-G_2)\right]T - \frac{3H\alpha}{4a^2(1+\theta^2)}M,$$

$$\delta = \frac{3H\alpha}{4a^2(1+\theta^2)}\left[-T + \frac{M}{a\theta\sqrt{\gamma_1\gamma_2}}\right]. \qquad (5.3.16)$$

The main problem. Now we may apply the reciprocal theorem in order to obtain the punch displacements due to a normal concentrated force N applied at the point $(\rho,0,z)$. The normal displacement of the punch is readily available from (8) as

$$w_{\mathrm{N}} = \frac{\mathrm{NH}\sinh(\pi\theta)}{\pi a\theta}\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}Y_c(x)\,\mathrm{d}x\right.$$

$$\left.-\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}Y_s(x)\,\mathrm{d}x\right\}. \qquad (5.3.17)$$

In order to find the tangential displacement of the punch, we have to apply a unit tangential force T in the positive Ox direction. From (11–14) and (16), one can find the stress functions as

$$f(x) = \frac{T\sqrt{\gamma_1\gamma_2}\cosh(\pi\theta)}{4\pi^2 a(1+\theta^2)}\left[3\frac{x}{a}Y_s(x) + \frac{1-2\theta^2}{\theta}Y_c(x)\right],$$

$$f_1(x) = \frac{T\sqrt{\gamma_1\gamma_2}\sinh(\pi\theta)}{4\pi^2 a(1+\theta^2)}\left[-3\frac{x}{a}Y_c(x) + \frac{1-2\theta^2}{\theta}Y_s(x)\right], \tag{5.3.18}$$

and the tangential displacements can be defined from (15) in the form

$$u_{\mathrm{N}} = \frac{NH\sqrt{\gamma_1\gamma_2}\sinh(\pi\theta)}{\pi a(1+\theta^2)\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\left(1\right.\right.\right.$$

$$\left.-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)\left(3\frac{x}{a}Y_s(x) + \frac{1-2\theta^2}{\theta}Y_c(x)\right)\mathrm{d}x$$

$$\left.\left.-\int_0^a\frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\left(-3\frac{x}{a}Y_c(x) + \frac{1-2\theta^2}{\theta}Y_s(x)\right)\mathrm{d}x\right]\right\}, \tag{5.3.19}$$

We need to apply to the punch a unit tilting moment M in order to find the angular displacement δ. The stress functions in this case are

$$f(x) = -\frac{3M\cosh(\pi\theta)}{4\pi^2 a^3\theta(1+\theta^2)}\left(xY_s(x) - \theta a Y_c(x)\right),$$

$$f_1(x) = \frac{3M\sinh(\pi\theta)}{4\pi^2 a^3\theta(1+\theta^2)}\left(xY_c(x) + \theta a Y_s(x)\right), \tag{5.3.20}$$

and the angular displacement will take the form

$$\delta_{\mathrm{N}} = \frac{3NH\sinh(\pi\theta)}{\pi a^3\theta(1+\theta^2)\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\left(xY_s(x) - \theta a Y_c(x)\right)\mathrm{d}x\right.\right.$$

$$\left.\left.-\int_0^a\frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\left(xY_c(x) + \theta a Y_s(x)\right)\mathrm{d}x\right]\right\}, \tag{5.3.21}$$

Formulae (17,19,21) are the main new results of this section.

It is of interest to compare the influence of a concentrated load on a bonded punch to that of a smooth punch. These two cases represent two extremes in the interaction between a punch and an elastic half-space, so the results give the upper and lower bounds for the parameters involved. The normal and angular displacements of a smooth punch due to a point load N applied at the point $(\rho,0,z)$ are (Fabrikant, 1989a)

$$w_{\rm N} = \frac{NH}{a}\sum_{k=1}^{2}\frac{m_k}{m_k-1}\sin^{-1}\left(\frac{a}{l_{2k}}\right), \tag{5.3.22}$$

$$\delta_{\rm N} = -\frac{3NH}{2a^3}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\rho\sin^{-1}\left(\frac{a}{l_{2k}}\right)-\frac{l_{1k}(l_{2k}^2-a^2)^{1/2}}{l_{2k}}\right]\right\}. \tag{5.3.23}$$

One should note that the solutions (17,21) and (22–23) coincide in the case of $\theta=0$. In the case of isotropy, this corresponds to the Poisson coefficient $\nu=1/2$. The greatest difference between the solutions for a bonded and a smooth punch is attained for the Poisson coefficient $\nu=0$. This value was used in numerical computations. Formulae (22–23) in the case of isotropy will take the form

$$w_{\rm N} = \frac{NH}{a}\left[\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{z(a^2-l_1^2)^{1/2}}{2(1-\nu)(l_2^2-l_1^2)}\right], \tag{5.3.24}$$

$$\delta_{\rm N} = \frac{3NH}{2a^3}\,\rho\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{a(l_2^2-a^2)^{1/2}}{l_2^2}+\frac{za^2(a^2-l_1^2)^{1/2}}{(1-\nu)l_2^2(l_2^2-l_1^2)}\right]. \tag{5.3.25}$$

The limiting cases of (17) and (21) in the case of isotropy are

$$w_{\rm N} = \frac{NH\sinh(\pi\theta)}{\pi a\theta}\left\{\int_0^a\left[\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.\right.$$

$$\left.-\frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{2(1-\nu)[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right]Y_c(x)\,{\rm d}x$$

$$-\frac{1}{2(1-\nu)}\int_0^a\left[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.$$

$$+\frac{z^2[x^2(2x^2+2z^2)-\rho^2)-l_1^4(x)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\Bigg]Y_s(x)\,\mathrm{d}x\Bigg]\Bigg\}, \tag{5.3.26}$$

$$\delta_{\mathrm{N}}=-\frac{3NH\sinh(\pi\theta)}{\pi a^3\theta(1+\theta^2)\rho}\Bigg\{\frac{1}{2(1-\nu)}\int_0^a\Bigg[(1-2\nu)\frac{l_2(x)[l_2^2(x)-\rho^2]^{1/2}}{l_2^2(x)-l_1^2(x)}$$

$$+\frac{z\rho^2[l_2^2(x)-x^2]^{1/2}}{[l_2^2(x)-l_1^2(x)]^3}[4x^2-3l_1^2(x)-l_2^2(x)]\Bigg][a\theta Y_c(x)-xY_s(x)]\,\mathrm{d}x$$

$$+\int_0^a\Bigg[\frac{l_1(x)[\rho^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}-\frac{z\rho^2[x^2-l_1^2(x)]^{1/2}}{2(1-\nu)[l_2^2(x)-l_1^2(x)]^3}[4x^2$$

$$-l_1^2(x)-3l_2^2(x)]\Bigg][xY_c(x)+a\theta Y_s(x)]\,\mathrm{d}x\Bigg\}. \tag{5.3.27}$$

The results of computation are presented in Fig. 5.3.2 (dimensionless parameter w_N/w^0 versus ρ/a) and Fig. 5.3.3 (the ratio δ_N/δ^0) for various $z/a=0, 0.1, 0.5, 1$. As before, the solid line curves give the results for a bonded punch, and the broken line curves give similar results for a smooth punch. The quantity $w^0=\pi NH/(2a)$ corresponds to the settlement of a smooth punch subjected to a central loading equal to N. The plot shows that the settlement of a smooth punch is always greater (up to about $0.1w^0$) than that of a bonded punch.

The parameter $\delta^0=3\pi NH/(4a^2)$ gives the maximum angle of inclination of a smooth punch. For the angle of inclination, the difference between the results for a smooth punch and a bonded punch does not exceed $0.08\delta^0$. All the computations were made for the Poisson coefficient $\nu=0$. Since in real materials $\nu>0$, the difference between the smooth and bonded punch solutions will be smaller than that indicated above.

5.4. Tangential loading underneath a smooth punch

The problem of a smooth circular punch penetrating a transversely isotropic elastic half-space and interacting with an arbitrarily located tangential concentrated load is considered. A closed form exact solution is obtained for the stress distribution under the punch as well as for the linear and angular displacements of the punch.

Fig. 5.3.2. Influence of bonding on the punch settlement

Fig. 5.3.3. Influence of bonding on angular inclination

The problem of interaction between a punch and an anchor load is of particular interest to geomechanics where the internal forces can be visualized as forces transmitted by anchoring regions located in the vicinity of structural foundation. Only some axisymmetric cases have been considered so far. The general case of interaction between a smooth circular punch and an arbitrarily located *tangenial* load is considered here for the first time. To the best of our knowledge, this problem was not considered before even for the case of isotropy.

Theory. We consider a transversely isotropic elastic half-space $z \geq 0$. A smooth punch of arbitrary base shape (Fig. 5.4.1) penetrates its boundary $z = 0$.

Fig. 5.4.1. Geometry of the problem

The domain of contact is a circle of radius a, with its centre coinciding with the coordinate system origin $\rho = 0$. Let a horizontal point force, with the components T_x and T_y, be applied at the point Q with the polar cylindrical coordinates (ρ, ϕ, z). We shall use its complex representation, namely, $T = T_x + iT_y$. We may assume, without loss of generality, that the system of external forces applied to the punch is such that eliminates the normal and angular displacements which otherwise would have been produced by the anchor load T. We need to find the normal tractions under the punch σ, the normal force N and the tilting moment M applied to the punch in order to compensate its displacements which otherwise would have been produced by the force T.

A general solution in terms of three harmonic functions to the mixed boundary value problem for a transversely isotropic elastic half-space was given in (Fabrikant, 1988c). One can deduce from the results of (Fabrikant, 1989a) that the normal displacements on the plane $z=0$ due to the force T can be defined as

$$w(\rho_0,\phi_0,0) = \frac{1}{2}H(T\bar{q} + \bar{T}q)\left[\frac{\gamma_1}{(m_1-1)R_1(R_1+z_1)} + \frac{\gamma_2}{(m_2-1)R_2(R_2+z_2)}\right], \tag{5.4.1}$$

where parameters H, γ_1, γ_2, m_1, and m_2 are defined in section 1, and

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}, \quad z_k = z/\gamma_k,$$

$$R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z_k^2]^{1/2}, \quad \text{for} \quad k=1,2. \tag{5.4.2}$$

The governing integral equation which relates the normal stress σ exerted by the punch to the displacement w, defined by (1), is as follows:

$$\int_0^{2\pi}\int_0^a \frac{\sigma(r,\psi)r\,dr\,d\psi}{[\rho_0^2 + r^2 - 2r\rho_0\cos(\phi_0 - \psi)]^{1/2}}$$

$$= -\frac{1}{2}(T\bar{q} + \bar{T}q)\left[\frac{\gamma_1}{(m_1-1)R_1(R_1+z_1)} + \frac{\gamma_2}{(m_2-1)R_2(R_2+z_2)}\right], \tag{5.4.3}$$

The general solution of the equation of the type (3) is given first in (Fabrikant, 1971a), but it is somewhat difficult to use directly due to the complexity of the right-hand side of (3). We can use a shortcut instead. In addition to the point $Q(\rho,\phi,z)$, let us introduce two more points, namely, $Q_0(\rho_0,\phi_0,0)$ and $S(r,\psi,0)$. The following identity can be easily verified:

$$\int_0^{2\pi}\int_0^a \frac{z}{R^3(Q,S)}\left[\frac{R(Q,S)}{h} + \tan^{-1}\left(\frac{h}{R(Q,S)}\right)\right]\frac{r\,dr\,d\psi}{R(Q_0,S)} = \frac{\pi^2}{R(Q,Q_0)}. \tag{5.4.4}$$

Here $R(\cdot,\cdot)$ stands for the distance between two points, and

$$h = (a^2 - r^2)^{1/2}(a^2 - l_1^2)^{1/2}/a.$$

The following notation is used throughout this section:

$$l_1(x,\rho,z) \equiv l_1(x) = \frac{1}{2}\{[(\rho + x)^2 + z^2]^{1/2} - [(\rho - x)^2 + z^2]^{1/2}\},$$

$$l_2(x,\rho,z) \equiv l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}+[(\rho-x)^2+z^2]^{1/2}\}. \tag{5.4.5}$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notations $l_{1k}(x)$ and $l_{2k}(x)$ are understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for k=1,2.

We can observe that the right-hand side of (3) can be obtained from the right-hand side of (4) by using integration with respect to z and the consequent application of the differentiation operator Λ defined by

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} . \tag{5.4.6}$$

A similar procedure, applied to the left-hand side of (4), will give us the required solution. The relevant integration and differentiation is performed in (Fabrikant, 1989a), and the result is

$$\sigma(r,\psi) = -\Re\left\{\frac{\bar{T}}{\pi^2}\left[\frac{\gamma_1 U(z_1)}{m_1-1}+\frac{\gamma_2 U(z_2)}{m_2-1}\right]\right\}, \tag{5.4.7}$$

where $\Re$ is the real part sign, the overbar everywhere indicates the complex conjugate value, and the following notation was introduced:

$$U(z) \equiv U(\rho,\phi,z;r,\psi) = \frac{q_1}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]-\frac{1}{h\bar{q}_1}\left\{1-\frac{(a^2-l_1^2)^{1/2}}{a}\right.$$
$$\left.-\frac{(a^2-l_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\left[\tan^{-1}\left(\frac{a(\bar{\zeta}-1)^{1/2}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}(\bar{\zeta}-1)^{1/2}\right]\right\}, \tag{5.4.8}$$

with

$$\zeta = \frac{\rho}{r}e^{i(\phi-\psi)}, \quad q_1 = \rho e^{i\phi} - re^{i\psi}, \quad R_0 = R(Q,S). \tag{5.4.9}$$

The resultant force N can be obtained by integration of σ

$$N = \int_0^{2\pi}\int_0^{a} \sigma(r,\psi)r\,dr\,d\psi. \tag{5.4.10}$$

Though expression (8) looks complicated, the integration in (10) can be performed (see Fabrikant, 1989a, for detail), and the result is rather simple:

$$N = -\Re\left\{\frac{2\overline{T}e^{i\phi}}{\pi\rho}\sum_{k=1}^{2}\left[\frac{\gamma_k[a-(a^2-l_{1k}^2)^{1/2}]}{m_k-1}\right]\right\}. \tag{5.4.11}$$

We define the tilting moment M in the complex form as follows:

$$M = M_x + iM_y = -i\int_0^{2\pi}\int_0^a \sigma(r,\psi)r^2 e^{i\psi}\mathrm{d}r\mathrm{d}\psi. \tag{5.4.12}$$

Again, the integration can be performed in elementary functions (see Fabrikant, 1989a), and the final result is

$$M = -\frac{2}{\pi}i\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k-1}\left[T\left(z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right)\right.\right.$$

$$\left.\left.-\overline{T}e^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right]\right\}. \tag{5.4.13}$$

Taking into consideration the relationship between the tilting moment and the complex angle of inclination

$$\delta = \delta_x + i\delta_y = \frac{3\pi H}{4a^3}M, \tag{5.4.14}$$

we may deduce that when no tilting moment is applied, the punch will tilt, and the angle will be

$$\delta = \frac{3H}{2a^3}i\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k-1}\left[T\left(z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right)\right.\right.$$

$$\left.\left.-\overline{T}e^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right]\right\}. \tag{5.4.15}$$

Formulae (7–9), (11), (13), and (15) are the main new results of this section.

Applications. The simplicity of the results obtained allows us to consider various cases of distributed internal loadings. Let a uniformly distributed shear tractions τ_0 be applied over the segment $\rho_1 \le \rho \le \rho_2$, $\phi_1 \le \phi \le \phi_2$, at the distance z from the surface. Integrating (11) by the method described in (Fabrikant, 1989a), we obtain

$$N = \Re\left\{\frac{2\mathrm{i}}{\pi}\tau_0(\mathrm{e}^{i\phi_2} - \mathrm{e}^{i\phi_1})\sum_{k=1}^{2}\left[\frac{\gamma_k}{m_k - 1}[V(\rho_2) - V(\rho_1)]\right]\right\}, \tag{5.4.16}$$

where

$$V(\rho) = \int [a - (a^2 - l_1^2)^{1/2}]\mathrm{d}\rho = a\rho - \frac{1}{2}\rho(a^2 - l_1^2)^{1/2}$$

$$-\frac{1}{2}(a^2 - z^2)\sin^{-1}\left(\frac{l_1}{(a^2 + z^2)^{1/2}}\right) - az\ln\frac{l_2 + (\rho^2 - l_1^2)^{1/2}}{(a^2 + z^2)^{1/2}}. \tag{5.4.17}$$

In the case of a uniform loading over a complete annulus, formula (16) gives N=0 as it was expected. The value of the tilting moment can be obtained by integration of (13), and we present the result for the case of uniform shear loading of a circular annulus $\rho_1 \le \rho \le \rho_2$

$$M = -\frac{2\mathrm{i}}{\pi}\tau_0\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k - 1}\left[W(\rho_2) - W(\rho_1)\right]\right\}. \tag{5.4.18}$$

Here

$$W(\rho) = \int \left[z\sin^{-1}\left(\frac{a}{l_2}\right) - (a^2 - l_1^2)^{1/2}\right]\rho\,\mathrm{d}\rho = \frac{1}{2}z\rho^2\sin^{-1}\left(\frac{a}{l_2}\right)$$

$$+\frac{z^2(2a^2 - l_1^2)}{2(a^2 - l_1^2)^{1/2}} - \frac{1}{3}(a^2 - l_1^2)^{1/2}(3l_2^2 + l_1^2 - 4a^2). \tag{5.4.19}$$

In the case of a torsional loading of a circular annulus, both the normal force and the tilting moment vanish.

The general solution is also valid in the case of isotropy, provided that we compute the limiting case

$$m_1 = m_2 = 1, \quad \gamma_1 = \gamma_2 = \gamma_3 = 1, \quad H = \frac{1 - \nu^2}{\pi E},$$

where E is the elastic modulus, and ν is Poisson coefficient.

According to the L'Hôpital rule, the following scheme should be used

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1 f(z_1)}{m_1-1}+\frac{\gamma_2 f(z_2)}{m_2-1}\right]=-\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \tag{5.4.20}$$

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1=1, \qquad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial\gamma_1}\right]=2(1-\nu), \tag{5.4.21}$$

and the symbol ($'$) indicates differentiation with respect to z. Formulae (7), (11), (13), and (15) in the case of isotropy will take the form

$$\sigma(r,\psi)=\Re\left\{\frac{\overline{T}}{2\pi^2(1-\nu)}\left[(1-2\nu)U(z)+zU'(z)\right]\right\}, \tag{5.4.22}$$

with $U(z)$ defined by (8–9)

$$U'(z)=\frac{\partial U}{\partial z}=-\frac{3zq}{R_0^5}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]+\frac{z}{h(R_0^2+h^2)}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}+\frac{q}{R_0^2}\right]; \tag{5.4.23}$$

$$N=\Re\left\{\frac{\overline{T}e^{i\phi}}{\pi\rho(1-\nu)}\left[(1-2\nu)[a-(a^2-l_1^2)^{1/2}]-\frac{zl_1(\rho^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right]\right\}, \tag{5.4.24}$$

$$M=\frac{\mathrm{i}}{\pi(1-\nu)}\left\{(1-2\nu)\left[T\left(z\sin^{-1}(\frac{a}{l_2})-(a^2-l_1^2)^{1/2}\right)\right.\right.$$

$$\left.-\overline{T}e^{2i\phi}\frac{2a^3-(l_1^2+2a^2)(a^2-l_1^2)^{1/2}}{3\rho^2}\right]+z\left[T\left(\sin^{-1}(\frac{a}{l_2})-\frac{a(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\right)\right.$$

$$\left.\left.+\overline{T}e^{2i\phi}\frac{al_1^2(l_2^2-a^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}\right]\right\}, \tag{5.4.25}$$

$$\delta=\frac{3\mathrm{i}}{8\pi\mu a^3}\left\{(1-2\nu)\left[T\left(z\sin^{-1}(\frac{a}{l_2})-(a^2-l_1^2)^{1/2}\right)\right.\right.$$

$$- \bar{T} e^{2i\phi} \frac{2a^3 - (l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2}}{3\rho^2} \Bigg] + z \Bigg[T \left(\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} \right)$$

$$+ \bar{T} e^{2i\phi} \frac{a l_1^2 (l_2^2 - a^2)^{1/2}}{l_2^2 (l_2^2 - l_1^2)} \Bigg] \Bigg\}, \tag{5.4.26}$$

In order to illustrate the stress distribution under the punch, computations were made due to (22–23), for $\nu=0.3$, $\phi=0$, $z/a=0.25$, $\rho/a=0.5$. The cases of a unit radial and a unit transversal loading are considered separately (Fig. 5.4.2 and Fig. 5.4.3). The curves given by the solid dots, the dashed line, the line of

Fig. 5.4.2. Stress distribution due to a unit radial force

circles, and the solid line correspond to the values of $\psi=0$, $\pi/4$, $\pi/2$, $3\pi/4$ respectively. The negative values of r correspond to the argument $\psi+\pi$. It is important to emphasize that the presented results are valid only when the contact is maintained all over the circle $r \le a$. Since the normal stress is negative at the part of the domain of contact, this implies that there should be additional external loading applied to the punch, so that the results stay valid.

Discussion. There exists an alternative way to solve the problem of interaction between an external load and a circular punch. Indeed, we could use the Green's functions derived in (Fabrikant, 1989a), combined with the reciprocal

Fig. 5.4.3. Stress distribution due to a unit transversal force

theorem. The tangential displacements around a circular punch are defined by

$$u(\rho,\phi,z) = \frac{1}{\pi^2}\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k - 1}\int_0^{2\pi}\int_0^{a} U(\rho,\phi,z_k; r,\psi)\,\omega(r,\psi)\,r\mathrm{d}r\mathrm{d}\psi\right\}. \tag{5.4.27}$$

Here U is defined by (8), and ω denotes the normal displacements under the punch. We consider two systems in equilibrium, namely, a tangential force is applied at the point (ρ,ϕ,z), and a normal stress σ is applied at the domain of contact in order to eliminate the normal displacements; the second system represents a transversely isotropic half-space, with a normal displacement prescribed at the point $(r,\psi,0)$ in the form $\omega=\delta(r,\psi)$, where δ is the Dirac delta-function. Application of the reciprocal theorem yields

$$\Re\{u\bar{T}\} + \int\int_S \sigma\,\omega\,\mathrm{d}S = 0. \tag{5.4.28}$$

Substitution of (27) in (28) and subsequent use of the properties of the delta-function lead immediately to (7).

The tangential displacement around a flat circular punch, subjected to a unit normal displacement, is (Fabrikant, 1989a)

$$u=\frac{2e^{i\phi}}{\pi\rho}\sum_{k=1}^{2}\left[\frac{\gamma_k[a-(a^2-l_{1k}^2)^{1/2}]}{m_k-1}\right]. \tag{5.4.29}$$

Again, application of the reciprocal theorem yields (11). In order to derive (26), we should use the expression for tangential displacement around an flat punch inclined by the angle $\delta=\delta_x+i\delta_y$ (Fabrikant, 1989a)

$$u=\frac{2}{\pi}i\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k-1}\left[\delta\left(z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right)\right.\right.$$

$$\left.\left.-\bar{\delta}e^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right]\right\}. \tag{5.4.30}$$

The derivation of (13) from (30) is straightforward.

5.5. The general annular punch problem

Quite a few papers have been published on the subject. One can find many references related to contact problem in (Barber, 1983), other references related to the equivalent electrostatic problem can be found in Love (1976). Why is there any need for yet another work on the subject? The main reason is that the majority of publications is devoted to the simplest flat centrally loaded annular punch problem. A very small number of publications treat non-flat but still axisymmetric problems (Barber 1976, 1983). Though some results related to consideration of specific harmonics have been published (Williams, 1963; Cooke, 1963), no general solution to the problem has been attempted as yet. Such a solution is presented here. The problem is reduced to a two-dimensional Fredholm integral equation with an elementary kernel which can be solved numerically. Flat inclined and centrally loaded annular punches are considered as examples. Asymptotic formulae are derived for the case of a very narrow annulus.

Theory. Consider a rigid annular punch $b\le\rho\le a$ penetrating a transversely isotropic elastic half space $z>0$. Neglecting the shear stress under the punch base, the boundary conditions for the problem can be formulated as follows:

$$w(\rho,\phi)=\delta-s(\rho,\phi),\quad \text{for}\quad b<\rho<a,\quad 0\le\phi<2\pi;$$

$$\sigma_z = 0, \quad \text{for} \quad \rho < b \quad \text{or} \quad \rho > a, \quad 0 \leq \phi < 2\pi;$$

$$\tau_{yz} = \tau_{zx} = 0, \quad \text{for} \quad 0 \leq \rho < \infty, \quad 0 \leq \phi < 2\pi. \tag{5.5.1}$$

Here δ is the maximum punch penetration and s describes the shape of the punch base. It is well known that the problem can be reduced to the governing integral equation

$$H \int_0^{2\pi} \int_b^a \frac{\sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi-\phi_0)}} = w(\rho,\phi). \tag{5.5.2}$$

Here H is the elastic constant (see 5.1.9), w is the known function (1), and $\sigma = -\sigma_z$ is the yet unknown function. The following integral representation for the reciprocal of the distance between two points can be found in (1.2.22)

$$\frac{1}{R} = \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)}} = \frac{2}{\pi} \int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, dx}{(\rho^2 - x^2)^{1/2} (\rho_0^2 - x^2)^{1/2}}. \tag{5.5.3}$$

Here

$$\lambda(k,\psi) = \frac{1 - k^2}{1 - 2k \cos\psi + k^2}. \tag{5.5.4}$$

Substitution of (3) in (2) leads to the governing integral equation

$$4 \int_b^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi)$$

$$+ 4 \int_0^b \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_b^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi) = \frac{w(\rho,\phi)}{H}. \tag{5.5.5}$$

The $\mathcal{L}$-operator for $k \leq 1$ was introduced as follows:

$$\mathcal{L}(k) f(\rho,\phi) = \frac{1}{2\pi} \int_0^{2\pi} \lambda(k, \phi-\phi_0)\, f(\rho,\phi_0)\, d\phi_0$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}k^{|n|}e^{in\phi}\int_0^{2\pi}e^{-in\phi_0}f(\rho,\phi_0)\,d\phi_0=\sum_{n=-\infty}^{\infty}k^{|n|}f_n(\rho)e^{in\phi}. \tag{5.5.6}$$

Here f_n is the n-th Fourier coefficient of the function f. Application of the operator

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_b^r\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)$$

to both sides of (5) yields

$$2\pi\int_r^a\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-r^2}}\mathcal{L}\left(\frac{r}{\rho_0}\right)\sigma(\rho_0,\phi)+\frac{4r}{\sqrt{r^2-b^2}}\int_0^b\frac{\sqrt{b^2-x^2}\,dx}{r^2-x^2}\int_b^a\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{x^2}{r\rho_0}\right)\sigma(\rho_0,\phi)$$

$$=\frac{1}{H}\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_b^r\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)w(\rho,\phi). \tag{5.5.7}$$

We introduce a new unknown function

$$\chi(r,\phi)=\int_r^a\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-r^2}}\mathcal{L}\left(\frac{r}{\rho_0}\right)\sigma(\rho_0,\phi). \tag{5.5.8}$$

The inverse of (8) is readily available, and is

$$\sigma(\rho,\phi)=-\frac{2}{\pi}\frac{\mathcal{L}(\rho)}{\rho}\frac{d}{d\rho}\int_\rho^a\frac{r\,dr}{\sqrt{r^2-\rho^2}}\mathcal{L}\left(\frac{1}{r}\right)\chi(r,\phi). \tag{5.5.9}$$

Substitution of (8) in (7) gives

$$2\pi\chi(r,\phi)+\frac{8}{\pi}\frac{r}{\sqrt{r^2-b^2}}\int_0^b\frac{b^2-x^2}{r^2-x^2}\,dx\int_b^a\frac{y\,dy}{\sqrt{y^2-b^2}(y^2-x^2)}\mathcal{L}\left(\frac{x^2}{yr}\right)\chi(y,\phi)$$

$$= \frac{1}{H}\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)w(\rho,\phi). \tag{5.5.10}$$

One can interchange the order of integration in the second term of (10) and perform the integration with respect to x. The result is

$$\chi(r,\phi) + \frac{1}{\pi^3}\int_0^{2\pi}\int_b^a \frac{K(y,r,\phi-\phi_0)-K(r,y,\phi-\phi_0)}{y^2-r^2}\chi(y,\phi_0)\,\mathrm{d}y\,\mathrm{d}\phi_0$$

$$= \frac{1}{2\pi H}\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)w(\rho,\phi). \tag{5.5.11}$$

The kernel of (11) can be expressed in terms of elementary functions as follows:

$$K(y,r,\phi-\phi_0) = ry\left(\frac{r^2-b^2}{y^2-b^2}\right)^{1/2}\left\{-\lambda\left(\frac{r}{y},\,\phi-\phi_0\right)\frac{1}{r}\ln\left(\frac{r+b}{r-b}\right)\right.$$

$$\left.+\,2\Re\left[\frac{1}{\xi\left(1-\frac{r}{y}e^{-i(\phi-\phi_0)}\right)}\ln\frac{\xi+b}{\xi-b}\right]\right\}, \tag{5.5.12}$$

where

$$\xi = \sqrt{yre^{i(\phi-\phi_0)}}. \tag{5.5.13}$$

Here $\Re$ denotes the real part of the expression to follow. Thus, the general problem of annular punch has been reduced to a Fredholm integral equation (11) with an elementary kernel which can be solved numerically. It is noteworthy that the governing equation for each specific harmonic will also have an elementary kernel. For example, the equation corresponding to the zero harmonic is

$$\chi_0(r) + \frac{2}{\pi^2}\int_b^a \frac{K_0(y,r)-K_0(r,y)}{y^2-r^2}\chi_0(y)\,\mathrm{d}y = \frac{1}{2\pi H}\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{w_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \tag{5.5.14}$$

with

$$K_0(y,r) = r\left(\frac{y^2-b^2}{r^2-b^2}\right)^{1/2} \ln\frac{y+b}{y-b}. \tag{5.5.15}$$

There have been so many variations of the governing integral equation published for the case of axial symmetry, that there is no doubt that equation (14) coincides with some known result, though we have difficulty to pinpoint exactly which one. The governing integral equation for the first harmonic will take the form

$$\chi_1(r) + \frac{2}{\pi^2}\int_b^a \frac{K_1(y,r)-K_1(r,y)}{y^2-r^2}\chi_1(y)\,\mathrm{d}y = \frac{1}{2\pi H}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{w_1(\rho)\rho^2\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \tag{5.5.16}$$

with

$$K_1(y,r) = \left(\frac{y^2-b^2}{r^2-b^2}\right)^{1/2}\left[y\ln\frac{y+b}{y-b} - 2b\right]. \tag{5.5.17}$$

There is no need to compute the stress distribution σ if one is interested in the integral characteristics only. Indeed, both the resultant force P and the tilting moment M can be expressed through the new unknown function χ as follows:

$$P = \frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{\chi(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{\rho^2-b^2}} = 4\int_b^a \frac{\chi_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}, \tag{5.5.18}$$

$$M = -\frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{(2\rho^2-b^2)\chi(\rho,\phi)\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{\rho^2-b^2}} = -2\int_b^a \frac{(2\rho^2-b^2)\chi_1(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}. \tag{5.5.19}$$

We note also that the kernels in (14) and (16) are finite at the point $y=r$. The following limits can be computed

$$\lim_{y\to r}\frac{K_0(y,r)-K_0(r,y)}{y^2-r^2} = \frac{1}{r^2-b^2}\left[\frac{r^2+b^2}{2r}\ln\frac{r+b}{r-b} - b\right],$$

$$\lim_{y\to r}\frac{K_1(y,r)-K_1(r,y)}{y^2-r^2} = \frac{1}{r^2-b^2}\left[\frac{3r^2-b^2}{2r}\ln\frac{r+b}{r-b} - 3b\right]. \tag{5.5.20}$$

Equations (11), (12), (18), and (19) are the main new results of this section.

Description of the numerical procedure. Consider the following integral equation:

$$h(r)\,f(r) + \int_b^a \mathcal{K}(r,x)\,f(x)\,\mathrm{d}x = g(r). \tag{5.5.21}$$

Here h and g are known functions, $\mathcal{K}$ is the kernel, and f is the as yet unknown function. The procedure which is usually used may be described as follows. We divide the interval $[b,a]$ into $n-1$ equal subintervals of length $\Delta=(a-b)/(n-1)$. The points of division are called x_k, $k=1,2,\ ...,n$. Assume the unknown function f to be piecewise *constant* on each of the subintervals and equal to f_k on the subinterval number k. Introduce a set of points $r_k=(x_k+x_{k+1})/2$, for $k=1,2,\ ...,n-1$. These assumptions allow us to reduce the integral equation (21) to a set of $n-1$ linear algebraic equations

$$h(r_k)\,f(r_k) + \sum_{i=1}^{n\text{-}1} \kappa_i(r_k)\,f_i = g(r_k), \quad \text{for} \quad k=1,2,...,n-1. \tag{5.5.22}$$

Here

$$\kappa_i(r_k) = \int_{x_i}^{x_{i+1}} \mathcal{K}(r_k,x)\,\mathrm{d}x. \tag{5.5.23}$$

The second method to be used here is somewhat different from that above. We consider the unknown function f to be piecewise *linear*. Assuming $f_k=f(x_k)$, for $k=1,2,\ ...,n$, this implies that at the k-th subinterval the function f can be expressed as follows:

$$f(x) = f_k + (f_{k+1}-f_k)\left(\frac{x-b}{\Delta}-k\right), \quad \text{for} \quad x_k < x < x_{k+1}. \tag{5.5.24}$$

Substitution of (24) in (21) leads to a set of n linear algebraic equations

$$h(r_l)f_l + f_1\left[\kappa_1(r_l) - \frac{\theta_1(r_l)}{\Delta}\right] + \sum_{i=2}^{n\text{-}1} f_i\left[i\kappa_i(r_l) - (i-2)\kappa_{i-1}(r_l) - \frac{\theta_i(r_l)-\theta_{i-1}(r_l)}{\Delta}\right]$$

$$+ f_n\left[\frac{\theta_{n-1}(r_l)}{\Delta} - (n-2)\kappa_{n-1}(r_l)\right] = g(r_l), \quad r_l = x_l, \quad \text{for} \quad l=1,2,...,n. \tag{5.5.25}$$

Here

$$\theta_i(r_l) = \int_{x_i}^{x_{i+1}} \mathcal{K}(r_l, x)\,(x-b)\,\mathrm{d}x. \tag{5.5.26}$$

Since the piecewise linear function follows the real function more close than the piecewise constant one, we should expect the set of equations (25) to give a more accurate solution than (22). One can also assume the function f to be piecewise *quadratic*. This exercise is left to the reader. Several examples are considered below.

Flat centrally loaded annular punch. In this case w_0=const., and the governing integral equation (14) will take the form

$$\chi_0(r) + \frac{2}{\pi^2}\int_b^a \frac{K_0(y,r) - K_0(r,y)}{y^2 - r^2}\,\chi_0(y)\,\mathrm{d}y = \frac{w_0}{2\pi H}\frac{r}{\sqrt{r^2-b^2}}. \tag{5.5.27}$$

It is well known that the stress distribution σ has square root singularities at $\rho = a$ and at $\rho = b$. We can then conclude from (8) that function χ_0 will have a logarithmic singularity at the point $\rho = b$. In order to obtain an effective numerical solution of (27) we have to eliminate singularities whenever possible. We introduce a new unknown function

$$f(r) = \frac{\chi_0(r)}{\ln\dfrac{r+b}{r-b}}, \tag{5.5.28}$$

which will have no singularities and will be limited on the $[b, a]$. Substitution of (28) in (27) allows us to rewrite it as follows:

$$\frac{\sqrt{r^2-b^2}}{r}\ln\frac{r+b}{r-b}f(r) + \frac{2}{\pi^2}\frac{\sqrt{r^2-b^2}}{r}\int_b^a \frac{K_0(y,r)-K_0(r,y)}{y^2-r^2}f(y)\ln\left(\frac{y+b}{y-b}\right)\mathrm{d}y = \frac{w_0}{2\pi H}. \tag{5.5.29}$$

Note that in the limiting case of $r \to b$ equation (29) yields

$$\int_b^a \ln^2\left(\frac{y+b}{y-b}\right)\frac{f(y)\,\mathrm{d}y}{\sqrt{y^2-b^2}} = \frac{\pi w_0}{4H}. \tag{5.5.30}$$

The problem was solved numerically by using both methods from the previous section. The value of the total force P was computed in the first method according to the formula (18) as follows:

$$P = 4\sum_{i=1}^{n-1} f_i \int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}. \tag{5.5.31}$$

The resultant force in the case of the second numerical method was computed as

$$P = 4\sum_{i=1}^{n-1}\left\{\left[\left(i+\frac{b}{\Delta}\right)f_i - \left(i+\frac{b}{\Delta}-1\right)f_{i+1}\right]\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}\right.$$

$$\left. + \frac{f_{i+1}-f_i}{\Delta}\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{\rho^2\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}\right\}. \tag{5.5.32}$$

The integrals in (31) and (32) can be computed exactly in terms of elementary functions or it can be computed numerically.

Numerical computations were performed according to both methods for different values of n and various ratios b/a. The dimensionless quantity $f^* = Hf/w_0$ is plotted in Fig. 5.5.2. versus $\rho^* = (\rho-b)/\Delta + 1$. The argument of each plot was scaled in such a way that it would stretch over the same interval. The dimensionless resultant force $P^* = P/P_0$ is presented in the Table 5.5.1. The quantity $P_0 = 2w_0 a/(\pi H)$ corresponds to the resultant force producing normal displacement w_0 when applied to a circular punch of radius a (see Fabrikant, 1989a, p. 342). The column denoted as exact was computed independently according to the formula derived in (Love, 1976). This formula in our notation reads

$$P^* = 1 - \sum_{n=1}^{\infty} k\int_0^k \left\{\int_0^k K_L^n(u,t)\frac{\mathrm{d}t}{t}\right\}^2 \mathrm{d}u. \tag{5.5.33}$$

Here $k = \sqrt{b/a}$, and K_L^n is the n-th iteration of the kernel

$$K_L(u,t) = \frac{2}{\pi}\frac{ut}{1-u^2t^2}. \tag{5.5.34}$$

Fig. 5.5.2. Solution for a centrally loaded annular punch

Let us point out some interesting features of the numerical results in Table 5.1.1. First of all, two different methods lead to different results, but the discrepancy between them decreases as *n* increases, and in such a way that their average changes very little being very close to the exact value. The second conclusion is that each of the methods gives either upper or lower bound for the computed quantity. This feature is extremely important since it allows us to estimate the error of computation. As we expected, the second method is everywhere more accurate than the first one.

An asymptotic solution for a very narrow annulus can be found by using the analogy with a two-dimensional contact problem. The stress distribution can be taken in the form

$$\sigma(\rho) = \frac{\sigma_0}{\sqrt{c^2 - (\rho - r_0)^2}}. \tag{5.5.35}$$

Here σ_0 is the as yet unknown constant,

$$c = (a - b)/2, \quad r_0 = (a + b)/2. \tag{5.5.36}$$

Substitution of (35) in (8) yields

Table 5.5.1.

n	b/a	Method 1 P^*	Method 2 P^*	Average P^*	Exact P^*
10	0.20000	1.0325	0.9855	1.0090	0.9989
	0.40000	1.0117	0.9807	0.9962	0.9907
	0.60000	0.9776	0.9587	0.9682	0.9651
	0.80000	0.9027	0.8950	0.8988	0.8976
	0.90000	0.8224	0.8204	0.8214	0.8210
	0.95000	0.7473	0.7483	0.7478	0.7478
	0.99500	0.5598	0.5632	0.5615	0.5618
	0.99950	0.4440	0.4471	0.4456	0.4458
	0.99995	0.3676	0.3702	0.3689	0.3691
20	0.20000	1.0119	0.9924	1.0022	0.9989
	0.40000	0.9993	0.9859	0.9926	0.9907
	0.60000	0.9704	0.9620	0.9662	0.9651
	0.80000	0.8998	0.8964	0.8981	0.8976
	0.90000	0.8216	0.8207	0.8212	0.8210
	0.95000	0.7475	0.7481	0.7478	0.7478
	0.99500	0.5608	0.5625	0.5617	0.5618
	0.99950	0.4449	0.4465	0.4457	0.4458
	0.99995	0.3684	0.3697	0.3690	0.3691
30	0.20000	1.0066	0.9946	1.0006	0.9989
	0.40000	0.9959	0.9876	0.9918	0.9907
	0.60000	0.9684	0.9631	0.9657	0.9651
	0.80000	0.8990	0.8968	0.8979	0.8976
	0.90000	0.8214	0.8208	0.8211	0.8210
	0.95000	0.7476	0.7480	0.7478	0.7478
	0.99500	0.5612	0.5623	0.5617	0.5618
	0.99950	0.4452	0.4463	0.4457	0.4458
	0.99995	0.3686	0.3695	0.3691	0.3691
40	0.20000	1.0043	0.9958	1.0000	0.9989
	0.40000	0.9944	0.9884	0.9914	0.9907
	0.60000	0.9674	0.9636	0.9655	0.9651
	0.80000	0.8986	0.8970	0.8978	0.8976
	0.90000	0.8213	0.8209	0.8211	0.8210
	0.95000	0.7477	0.7479	0.7478	0.7478
	0.99500	0.5613	0.5622	0.5618	0.5618
	0.99950	0.4454	0.4462	0.4458	0.4458
	0.99995	0.3687	0.3694	0.3691	0.3691

$$\chi(t) \approx \sigma_0 \frac{\sqrt{r_0}}{\sqrt{2}} \int_t^c \frac{\mathrm{d}x}{\sqrt{c^2 - x^2}\sqrt{x - t}}. \tag{5.5.37}$$

Here the following new variables were introduced

$$\rho_0 = r_0 + x, \quad r = r_0 + t, \tag{5.5.38}$$

and the small quantities of the order of c/r_0, x/r_0, and t/r_0 were neglected. The same procedure applied to (18) and (30) yields respectively

$$P = 2\sigma_0 \sqrt{2r_0} \int_{-c}^{c} \frac{\chi(t)\,\mathrm{d}t}{\sqrt{t + c}}, \tag{5.5.39}$$

$$w_0 = \frac{2\sqrt{2}\sigma_0}{\pi H \sqrt{r_0}} \int_{-c}^{c} \ln\left(\frac{2r_0}{t + c}\right) \frac{\chi(t)\,\mathrm{d}t}{\sqrt{t + c}}, \tag{5.5.40}$$

Substitution of (37) in (39) and (40), interchange of the order of integration and subsequent integration yield

$$P = 2\pi^2 r_0 \sigma_0, \tag{5.5.41}$$

$$w_0 = 2\pi H \sigma_0 \ln\left(\frac{16r_0}{c}\right). \tag{5.5.42}$$

Here the following integral was used

$$\int_{-c}^{x} \ln\left(\frac{2r_0}{t + c}\right) \frac{\mathrm{d}t}{\sqrt{t + c}\sqrt{x - t}} = \pi \ln\left(\frac{8r_0}{x + c}\right). \tag{5.5.43}$$

We may now deduce from (41) and (42) that

$$P = \frac{\pi w_0 r_0}{H \ln\left(\frac{16r_0}{c}\right)}, \tag{5.5.44}$$

$$\sigma_0 = \frac{P}{2\pi^2 r_0} = \frac{w_0}{2\pi H \ln\left(\frac{16 r_0}{c}\right)}, \tag{5.5.45}$$

$$P^* = \frac{\pi^2}{2\ln\left(\frac{16(a+b)}{a-b}\right)}. \tag{5.5.46}$$

The last result is in agreement with that of Smythe (1951) and Collins (1963).

Flat inclined annular punch. Assume that the punch is tilted about axis Oy in the positive direction, and that the angle of rotation is α. The normal displacements under the punch can be expressed as

$$w(\rho,\phi) = -\alpha\rho\cos\phi. \tag{5.5.47}$$

Substitution of (47) in (16) leads to the governing integral equation

$$\chi_1(r) + \frac{2}{\pi^2}\int_b^a \frac{K_1(y,r) - K_1(r,y)}{y^2 - r^2}\chi_1(y)\,\mathrm{d}y = -\frac{\alpha}{2\pi H}\frac{2r^2 - b^2}{\sqrt{r^2 - b^2}}. \tag{5.5.48}$$

It is reminded that the kernel K_1 is defined by (17). We may conclude once again that since the stress distribution is singular at the edges $\rho = b$ and $\rho = a$, the function χ_1 will have a logarithmic singularity at the point $\rho = b$. Introducing a new unknown function q as

$$q(r) = \frac{\chi_1(r)}{\ln\frac{r+b}{r-b}}, \tag{5.5.49}$$

we may rewrite (48) in the form

$$\frac{\sqrt{r^2-b^2}}{2r^2-b^2}\ln\frac{r+b}{r-b}\,q(r) + \frac{2}{\pi^2}\frac{\sqrt{r^2-b^2}}{2r^2-b^2}\int_b^a \frac{K_1(y,r) - K_1(r,y)}{y^2-r^2}q(y)\ln\left(\frac{y+b}{y-b}\right)\mathrm{d}y = -\frac{\alpha}{2\pi H}. \tag{5.5.50}$$

The problem was solved numerically by using both methods from the previous section. The value of the tilting moment M was computed in the first method according to the formula (19) as follows:

$$M = -2\sum_{i=1}^{n-1} q_i \int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,d\rho}{\sqrt{\rho^2-b^2}}. \tag{5.5.51}$$

The following formula was used in the second numerical method

$$M = -2\sum_{i=1}^{n-1}\left\{\left[q_i\left(i+\frac{b}{\Delta}\right)-q_{i+1}\left(i+\frac{b}{\Delta}-1\right)\right]\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,d\rho}{\sqrt{\rho^2-b^2}}\right.$$

$$\left.+\frac{q_{i+1}-q_i}{\Delta}\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,\rho\,d\rho}{\sqrt{\rho^2-b^2}}\right\}. \tag{5.5.52}$$

The integrals in (51) and (52) can be computed in terms of elementary functions, namely,

$$\int \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,d\rho}{\sqrt{\rho^2-b^2}} = \sqrt{\rho^2-b^2}\left[2b+\rho\ln\left(\frac{\rho+b}{\rho-b}\right)\right],$$

$$\int \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,\rho\,d\rho}{\sqrt{\rho^2-b^2}} = \frac{1}{3}(2\rho^2+b^2)\sqrt{\rho^2-b^2}\,\ln\left(\frac{\rho+b}{\rho-b}\right)$$

$$+\frac{2}{3}b\left(\rho\sqrt{\rho^2-b^2}\ +2b^2\ln(\rho+\sqrt{\rho^2-b^2}\right),$$

Numerical computations were performed according to both methods for different values of n and various ratios b/a. The dimensionless quantity $q^* = H|f|/(a\alpha)$ is plotted in Fig. 5.5.3. versus $\rho^* = (\rho - b)/\Delta + 1$. The conventions are the same as in the Fig. 5.5.2. The dimensionless tilting moment $M^* = M/M_0$ is presented in the Table 5.5.2. The quantity $M_0 = 4a^3\alpha/(3\pi H)$ corresponds to the tilting moment producing angular displacement α when applied to a circular punch of radius a. Here we no longer have that peculiar property that each method gives either upper or lower bound for the solution. It does not hold for $b/a = 0.2$, though it appears to be valid for $b/a \geq 0.4$.

An asymptotic solution for a very narrow $[(a-b)/a] \ll 1$ annulus can be attempted as above. Assume

Fig. 5.5.3. Solution for an inclined annular punch

$$\sigma(\rho,\phi)=\frac{\sigma_1\cos\phi}{\sqrt{c^2-(\rho-r_0)^2}}. \tag{5.5.53}$$

Here, as before, $2c$ is the annulus thickness and r_0 is its average radius as defined in (36). Substitution of (53) in (8) yields

$$\chi_1(t)\approx\sigma_1\frac{\sqrt{r_0}}{\sqrt{2}}\int_t^c\frac{\mathrm{d}x}{\sqrt{c^2-x^2}\sqrt{x-t}}. \tag{5.5.54}$$

Here the new variables were introduced in the manner similar to (38). In the limiting case of $r\to b$ we can deduce from (50) that

$$\alpha=-\frac{4H}{\pi b^2}\int_b^a\left[y\ln\left(\frac{y+b}{y-b}\right)-2b\right]\frac{\chi_1(y)\,\mathrm{d}y}{\sqrt{y^2-b^2}}\approx-\frac{2\sqrt{2}H}{\pi b^2\sqrt{r_0}}\int_{-c}^{c}\left[r_0\ln\left(\frac{2r_0}{t+c}\right)-2b\right]\frac{\chi_1(t)\,\mathrm{d}t}{\sqrt{t+c}}. \tag{5.5.55}$$

Substitution of (54) in (55) yields after interchanging the order of integration and subsequent integration

$$\alpha=-\frac{2\pi H\sigma_1}{r_0}\left[\ln\left(\frac{16r_0}{c}\right)-2\right]. \tag{5.5.56}$$

Table 5.5.2.

n	b/a	Method 1 M^*	Method 2 M^*	Average M^*	Exact M^*
10	0.2000	1.0028	1.0017	1.0023	0.99996
	0.4000	1.0079	0.9959	1.0019	0.99878
	0.6000	1.0012	0.9839	0.9925	0.98930
	0.8000	0.9491	0.9368	0.9429	0.94084
	0.9000	0.8658	0.8609	0.8633	0.86243
	0.9400	0.7991	0.7982	0.7986	0.79830
	0.9800	0.6677	0.6707	0.6692	0.66942
	0.9900	0.5989	0.6027	0.6008	0.60115
	0.9990	0.4392	0.4429	0.4411	0.44138
	0.9999	0.3449	0.3479	0.3464	0.34661
20	0.2000	1.0012	1.0000	1.0006	0.99996
	0.4000	1.0024	0.9971	0.9997	0.99878
	0.6000	0.9941	0.9866	0.9904	0.98930
	0.8000	0.9443	0.9389	0.9416	0.94084
	0.9000	0.8638	0.8617	0.8627	0.86243
	0.9400	0.7986	0.7982	0.7984	0.79830
	0.9800	0.6685	0.6701	0.6693	0.66942
	0.9900	0.6000	0.6020	0.6010	0.60115
	0.9990	0.4403	0.4422	0.4412	0.44138
	0.9999	0.3458	0.3473	0.3465	0.34661
30	0.2000	1.0007	0.9998	1.0003	0.99996
	0.4000	1.0010	0.9976	0.9993	0.99878
	0.6000	0.9923	0.9875	0.9899	0.98930
	0.8000	0.9430	0.9396	0.9413	0.94084
	0.9000	0.8633	0.8620	0.8626	0.86243
	0.9400	0.7984	0.7983	0.7983	0.79830
	0.9800	0.6688	0.6699	0.6693	0.66942
	0.9900	0.6004	0.6017	0.6010	0.60115
	0.9990	0.4407	0.4419	0.4413	0.44138
	0.9999	0.3461	0.3471	0.3466	0.34661
40	0.2000	1.0005	0.9998	1.0002	0.99996
	0.4000	1.0003	0.9979	0.9991	0.99878
	0.6000	0.9914	0.9880	0.9897	0.98930
	0.8000	0.9424	0.9399	0.9411	0.94084
	0.9000	0.8630	0.8621	0.8625	0.86243
	0.9400	0.7984	0.7983	0.7983	0.79830
	0.9800	0.6690	0.6697	0.6694	0.66942
	0.9900	0.6006	0.6016	0.6011	0.60115
	0.9990	0.4408	0.4418	0.4413	0.44138
	0.9999	0.3462	0.3470	0.3466	0.34661

A similar procedure performed on $(0\sqrt{t + c}]$.

(5.5.57)

Substitution of (54) in (57) yields after interchanging the order of integration and subsequent integration

$$\alpha = -\frac{2\pi H\sigma_1}{r_0}\left[\ln\left(\frac{16r_0}{c}\right) - 2\right]. \tag{5.5.58}$$

A similar procedure performed on $(42 - 2\Big]$. (5.5.59)

Taking into consideration that for a circular punch of radius a we have

$$M_0 = \frac{4a^3\alpha}{3\pi H}, \tag{5.5.60}$$

the following expression for the dimensionless moment can be written

$$M^* = \frac{M}{M_0} = \frac{3\pi^2(a+b)^3}{64a^3\left[\ln\left(\frac{16(a+b}{a-b}\right) - 2\right]} \tag{5.5.61}$$

We are unaware of a similar result published elsewhere.

Discussion. An attempt can be made to obtain an approximate analytical solution. We can multiply both sides of (27) by $4r\,\mathrm{d}r/\sqrt{r^2-b^2}$ and integrate with respect to r from $b+\varepsilon$ to a. The result is

$$4\int_{b+\varepsilon}^{a}\frac{\chi_0(r)\,r\,\mathrm{d}r}{\sqrt{r^2-b^2}} + \frac{4}{\pi^2}\int_{b}^{a}\frac{1}{\sqrt{y^2-b^2}}\Bigg\{\ln\left(\frac{y+b}{y-b}\right)\left[y\ln\frac{(y-b)(a+y)}{(y+b)(a-y)} - b\ln\frac{(a+b)\varepsilon}{(a-b)2b}\right.$$

$$\left. - y\int_{b}^{a}\ln\left(\frac{x+b}{x-b}\right)\frac{x\,\mathrm{d}x}{y^2-x^2}\Bigg\}\chi_0(y)\,\mathrm{d}y = \frac{2w_0}{\pi H}\left[a - b - \frac{b}{2}\ln\frac{(a+b)\varepsilon}{(a-b)2b}\right]. \tag{5.5.62}$$

By using identity (30) the limiting case of $\varepsilon \to 0$ can be computed

$$P + \frac{4}{\pi^2}\int_{b}^{a}T(y)\,\frac{\chi_0(y)\,y\,\mathrm{d}y}{\sqrt{y^2-b^2}} = \frac{2w_0(a-b)}{\pi H}, \tag{5.5.63}$$

with

$$T(y)=\ln\left(\frac{y+b}{y-b}\right)\ln\left(\frac{(a+y)(y-b)}{(a-y)(y+b)}\right)-\int_b^a \ln\left(\frac{x+b}{x-b}\right)\frac{x\,dx}{y^2-x^2}. \tag{5.5.64}$$

Taking into consideration that χ_0 does not change sign in the interval $[b,a]$, we can use the mean value theorem and to rewrite (63) as

$$P\left[1+\frac{1}{\pi^2}T(Y)\right]=\frac{2w_0(a-b)}{\pi H},$$

with an immediate consequence

$$P=\frac{2w_0(a-b)}{\pi H\left(1+\frac{1}{\pi^2}T(Y)\right)}. \tag{5.5.65}$$

We know about the value of Y only that it is located somewhere in the interval $[b,a]$. This condition allows infinite variation of T, thus making (65) of little practical value. On the other hand, formula (65) is exact in two limiting cases, namely for $b\to 0$ and $b\to a$. this means that an additional investigation can reveal an optimal value of Y, making (65) useful.

Yet another solution can be deduced from (18) and (30) which can be rewritten as

$$\int_b^a \ln\left(\frac{y+b}{y-b}\right)\frac{\chi_0(y)\,dy}{\sqrt{y^2-b^2}}=\frac{\pi w_0}{4H}. \tag{5.5.66}$$

Taking into consideration that χ_0 does not change sign in the $[b,a]$, we can apply the mean value theorem to (66), with the result

$$\frac{1}{Y}\ln\left(\frac{Y+b}{Y-b}\right)\int_b^a \frac{\chi_0(y)\,y\,dy}{\sqrt{y^2-b^2}}=\frac{\pi w_0}{4H}. \tag{5.5.67}$$

Comparison of (67) with (18) yields

$$P = \frac{\pi w_0 Y}{H \ln\left(\frac{Y+b}{Y-b}\right)}. \tag{5.5.68}$$

Again, the main problem with (68) is the fact that we know about the value of Y only that it is located somewhere between b and a, which allows infinite variation. This does not preclude, however, from finding some optimal value for Y which would make (68) useful. This investigation is beyond the scope of this book.

The complete solution, namely, the explicit expressions for the field of stresses and displacements in the elastic half-space due to the annular punch indentation, can be derived in terms of the function χ. Indeed, in order to obtain the field of normal displacements, one has to compute the integral

$$I(\rho,\phi,z) = \int_0^{2\pi}\int_b^a \frac{\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2}}. \tag{5.5.69}$$

Substitution of (9) in (69) yields, after interchanging the order of integration and subsequent integration

$$I(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{[l_2^2(x)-x^2]^{1/2}\,\chi(x,\phi_0)\,dx\,d\phi_0}{\rho^2+x^2-2\rho x\cos(\phi-\phi_0)+z^2}$$

$$+\frac{8}{\pi}\int_b^a \left\{\int_0^{l_1(b)} \mathcal{L}\left(\frac{x^2}{\rho y}\right)\frac{\sqrt{b^2-g^2(x)}\,dx}{[y^2-g^2(x)]\sqrt{\rho^2-x^2}}\right\}\frac{\chi(y,\phi)\,y\,dy}{\sqrt{y^2-b^2}}. \tag{5.5.70}$$

Here

$$l_1(x) = \frac{1}{2}\{\sqrt{(\rho+x)^2+z^2}-\sqrt{(\rho-x)^2+z^2}\},$$

$$l_2(x) = \frac{1}{2}\{\sqrt{(\rho+x)^2+z^2}+\sqrt{(\rho-x)^2+z^2}\},$$

$$g(x) = x\left[1+\frac{z^2}{\rho^2-x^2}\right]^{1/2}. \tag{5.5.71}$$

Note that function g is inverse to both l_1 and l_2. The method of integration is described in (Fabrikant, 1989a). The reader may now try to find a complete solution. One can also apply to the problem of an annular punch the results of sections 3.6 and 3.8.

Appendix 5

We present here some mathematical results used in various transformations throughout this book. The main properties of l_1 and l_2 are:

$$l_1 l_2 = a\rho, \quad l_1^2 + l_2^2 = a^2 + \rho^2 + z^2, \tag{A5.1}$$

$$(l_2^2-\rho^2)^{1/2}(l_2^2-a^2)^{1/2} = z l_2, \quad (a^2-l_1^2)^{1/2}(\rho^2-l_1^2)^{1/2} = z l_1,$$

$$(a^2-l_1^2)^{1/2}(l_2^2-a^2)^{1/2} = za, \quad (l_2^2-\rho^2)^{1/2}(\rho^2-l_1^2)^{1/2} = z\rho. \tag{A5.2}$$

$$\frac{\partial l_1}{\partial z} = -\frac{z l_1}{l_2^2-l_1^2}, \quad \frac{\partial l_2}{\partial z} = \frac{z l_2}{l_2^2-l_1^2},$$

$$\frac{\partial l_1}{\partial \rho} = \frac{a l_2 - \rho l_1}{l_2^2-l_1^2} = \frac{\rho(a^2-l_1^2)}{l_1(l_2^2-l_1^2)}, \quad \frac{\partial l_2}{\partial \rho} = \frac{\rho l_2 - a l_1}{l_2^2-l_1^2} = \frac{\rho(l_2^2-a^2)}{l_2(l_2^2-l_1^2)}. \tag{A5.3}$$

Here are some derivatives used

$$\frac{\partial}{\partial z}(l_2^2-a^2)^{1/2} = \frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.4}$$

$$\Lambda(l_2^2-a^2)^{1/2} = \frac{\rho e^{i\phi}(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.5}$$

$$\Lambda\frac{\partial}{\partial z}(l_2^2-a^2)^{1/2} = \frac{z[a^2(2a^2+2z^2-\rho^2)-l_2^4]}{(l_2^2-a^2)^{1/2}(l_2^2-l_1^2)^3}\rho e^{i\phi}, \tag{A5.6}$$

$$\frac{\partial^2}{\partial z^2}(l_2^2-a^2)^{1/2} = \frac{a^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-l_1^2)^3}(l_2^2+3l_1^2-4\rho^2), \tag{A5.7}$$

$$\frac{\partial}{\partial z}(a^2-l_1^2)^{1/2} = \frac{l_1(\rho^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.8}$$

$$\Lambda(a^2-l_1^2)^{1/2} = -\frac{\rho e^{i\phi}(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.9}$$

$$\Lambda\frac{\partial}{\partial z}(a^2-l_1^2)^{1/2} = -\frac{\rho e^{i\phi} z[l_1^4-a^2(2a^2+2z^2-\rho^2)]}{(a^2-l_1^2)^{1/2}(l_2^2-l_1^2)^3}, \tag{A5.10}$$

$$\frac{\partial^2}{\partial z^2}(a^2-l_1^2)^{1/2} = \frac{a^2(\rho^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)^3}(4\rho^2-l_1^2-3l_2^2). \tag{A5.11}$$

$$\frac{\partial}{\partial a}(a^2-l_1^2)^{1/2} = \frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2} = \frac{\partial}{\partial z}(l_2^2-a^2)^{1/2}, \tag{A5.12}$$

$$\frac{\partial}{\partial a}\sin^{-1}(\frac{a}{l_2}) = \frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}, \quad \frac{\partial}{\partial a}\ln[l_2+(l_2^2-\rho^2)^{1/2}] = \frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.13}$$

$$\frac{\partial}{\partial z}\sin^{-1}(\frac{a}{l_2}) = -\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \quad \frac{\partial}{\partial z}\ln[l_2+(l_2^2-\rho^2)^{1/2}] = \frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.14}$$

$$\Lambda\sin^{-1}(\frac{a}{l_2}) = -\frac{l_1(l_2^2-a^2)^{1/2}}{l_2[l_2^2-l_1^2]}\, e^{i\phi}, \tag{A5.15}$$

$$\Lambda^2\sin^{-1}(\frac{a}{l_2}) = \frac{a e^{2i\phi}(l_2^2-a^2)^{1/2}}{l_2^2[l_2^2-l_1^2]^3}\,[3\rho^2 l_2^2+\rho^2 l_1^2-6a^2\rho^2+2l_1^4] \tag{A5.16}$$

$$\Lambda\ln[l_2+(l_2^2-\rho^2)^{1/2}] = \frac{e^{i\phi}}{\rho}\left[1-\frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2}\right], \tag{A5.17}$$

$$\Lambda^2\ln[l_2+(l_2^2-\rho^2)^{1/2}] = -\frac{2e^{2i\phi}}{\rho^2} - \frac{a e^{2i\phi}(a^2-l_1^2)^{1/2}}{l_1^2(l_2^2-l_1^2)^3}\,[6a^2\rho^2-2l_2^4-\rho^2 l_2^2-3\rho^2 l_1^2], \tag{A5.18}$$

$$\frac{\partial^2}{\partial z^2}\sin^{-1}(\frac{a}{l_2}) = -\frac{\partial}{\partial z}\left(\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right) = \frac{z[a^2(2a^2+2z^2-\rho^2)-l_1^4]}{(a^2-l_1^2)^{1/2}(l_2^2-l_1^2)^3}, \tag{A5.19}$$

$$\frac{\partial^2}{\partial z^2}\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] = \frac{\partial}{\partial z}\left(\frac{(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}\right) = \frac{z[a^2(2a^2 + 2z^2 - \rho^2) - l_2^4]}{(l_2^2 - a^2)^{1/2}(l_2^2 - l_1^2)^3}, \qquad \text{(A5.20)}$$

$$\Lambda\frac{\partial}{\partial z}\sin^{-1}\left(\frac{a}{l_2}\right) = -\Lambda\left(\frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2}\right) = \frac{\rho e^{i\phi}(a^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)^3}[3l_2^2 + l_1^2 - 4a^2], \qquad \text{(A5.21)}$$

$$\Lambda\frac{\partial}{\partial z}\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] = \Lambda\left(\frac{(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}\right) = \frac{\rho e^{i\phi}(l_2^2 - a^2)^{1/2}}{(l_2^2 - l_1^2)^3}[4a^2 - l_2^2 - 3l_1^2]. \qquad \text{(A5.22)}$$

The following indefinite integrals are used:

$$\int (l_2^2 - a^2)^{1/2} da = (l_2^2 - a^2)^{1/2}\left(a - \frac{l_1^2}{2a}\right) + \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right), \qquad \text{(A5.23)}$$

$$\int (a^2 - l_1^2)^{1/2} da = -(a^2 - l_1^2)^{1/2}\left(\frac{l_2^2}{2a} - a\right) - \frac{\rho^2}{2}\ln\frac{a + (a^2 - l_1^2)^{1/2}}{l_1}, \qquad \text{(A5.24)}$$

$$\int (a^2 - l_1^2)^{1/2} a\, da = \frac{1}{6}(a^2 - l_1^2)^{1/2}(2a^2 - 3\rho^2 + l_1^2) + \frac{1}{2}z\rho^2\sin^{-1}\left(\frac{a}{l_2}\right), \qquad \text{(A5.25)}$$

$$\int (l_2^2 - a^2)^{1/2} dz = (a^2 - l_1^2)^{1/2}\,\frac{l_2^2 - 2a^2}{2a} + \frac{\rho^2}{2}\ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \qquad \text{(A5.26)}$$

$$\int (l_2^2 - a^2)^{1/2}\, l_1^2 dz = -a(a^2 - l_1^2)^{1/2}\,\frac{l_1^2 + 2a^2}{3} + a^2\rho^2\ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \qquad \text{(A5.27)}$$

$$\int (a^2 - l_1^2)^{1/2} dz = \frac{2a^2 - l_1^2}{2a}(l_2^2 - a^2)^{1/2} + \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right), \qquad \text{(A5.28)}$$

$$\int (a^2 - l_1^2)^{1/2}\, l_1^2 dz = -\frac{l_1^2(2l_1^2 + 3\rho^2)}{8a}(l_2^2 - a^2)^{1/2} + \rho^2\left(\frac{3}{8}\rho^2 - a^2\right)\sin^{-1}\left(\frac{a}{l_2}\right), \qquad \text{(A5.29)}$$

$$\int l_2^2(a^2 - l_1^2)^{1/2} dz = \frac{1}{3}a(l_2^2 - a^2)^{1/2}(2a^2 + l_2^2) + a^2\rho^2\sin^{-1}\left(\frac{a}{l_2}\right), \qquad \text{(A5.30)}$$

$$\int (l_2^2-a^2)^{1/2}\frac{l_1^2}{l_2^2}\,\mathrm{dz} = a(a^2-l_1^2)^{1/2}\left[1-\frac{8a^2}{15\rho^2}-\frac{4a^2+3l_1^2}{15l_2^2}\right], \tag{A5.31}$$

$$\int \frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\,\mathrm{dz} = -\sin^{-1}(\frac{a}{l_2}), \tag{A5.32}$$

$$\int \frac{(a^2-l_1^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}\,\mathrm{dz} = \frac{1}{2a^2}\left[\frac{a(l_2^2-a^2)^{1/2}}{l_2^2}-\sin^{-1}(\frac{a}{l_2})\right], \tag{A5.33}$$

$$\int \sin^{-1}(\frac{a}{l_2})\,\mathrm{dz} = \mathrm{z}\sin^{-1}(\frac{a}{l_2})-(a^2-l_1^2)^{1/2}+a\ln[l_2+(l_2^2-\rho^2)^{1/2}], \tag{A5.34}$$

$$\int \mathrm{z}\sin^{-1}(\frac{a}{l_2})\,\mathrm{dz} = \frac{1}{4}(2a^2+2z^2+\rho^2)\sin^{-1}(\frac{a}{l_2})+(l_2^2-a^2)^{1/2}\frac{2a^2+l_1^2}{4a}, \tag{A5.35}$$

$$\int z^2\sin^{-1}(\frac{a}{l_2})\,\mathrm{dz} = \frac{1}{3}z^3\sin^{-1}(\frac{a}{l_2})+\frac{1}{18}(a^2-l_1^2)^{1/2}(3l_2^2+6\rho^2+8a^2-2l_1^2)$$
$$-\frac{1}{6}a\,(3\rho^2+2a^2)\ln[l_2+(l_2^2-\rho^2)^{1/2}]. \tag{A5.36}$$

The following integrals may be computed by the method described in (Fabrikant, 1989a)

$$I = \int_0^{2\pi}\int_0^{a}\frac{z\ln(R_0+z)-R_0}{(a^2-\rho_0^2)^{1/2}}\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0 = \pi\left\{\left(z^2-a^2-\frac{\rho^2}{2}\right)\sin^{-1}(\frac{a}{l_2})\right.$$
$$\left.-\frac{3(2a^2-l_1^2)}{2a}(l_2^2-a^2)^{1/2}+2az\ln[l_2+(l_2^2-\rho^2)^{1/2}]\right\}, \tag{A5.37}$$

We recall that R_0 is defined by (5.1.16), and the parameters l_1 and l_2 are given by (5.1.33). Here are some partial derivatives used in this Chapter

$$\frac{\partial I}{\partial z} = \int_0^{2\pi}\int_0^{a}\frac{\ln(R_0+z)}{(a^2-\rho_0^2)^{1/2}}\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0$$

$$= 2\pi\left\{ z\sin^{-1}(\frac{a}{l_2}) - (a^2 - l_1^2)^{1/2} + a\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] \right\}, \tag{A5.38}$$

$$\frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{z\ln(R_0 + z) - R_0}{(a^2 - \rho_0^2)^{1/2}}\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0$$

$$= 2\pi\left\{ z\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] - a\sin^{-1}(\frac{a}{l_2}) - (l_2^2 - a^2)^{1/2} \right\}, \tag{A5.39}$$

$$\Lambda I = -\int_0^{2\pi}\int_0^a \frac{q\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{(R_0 + z)(a^2 - \rho_0^2)^{1/2}}$$

$$= -2\pi\frac{\mathrm{e}^{i\phi}}{\rho}\left[(l_2^2 - a^2)^{1/2}\left(a - \frac{l_1^2}{2a} \right) - za + \frac{\rho^2}{2}\sin^{-1}(\frac{a}{l_2}) \right], \tag{A5.40}$$

$$\frac{\partial^2 I}{\partial z^2} = \int_0^{2\pi}\int_0^a \frac{\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{R_0(a^2 - \rho_0^2)^{1/2}} = 2\pi\sin^{-1}(\frac{a}{l_2}) \tag{A5.41}$$

$$\frac{\partial^2 I}{\partial z \partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{\ln(R_0 + z)}{(a^2 - \rho_0^2)^{1/2}}\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0 = 2\pi\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] \tag{A5.42}$$

$$\Lambda\frac{\partial I}{\partial a} = -\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{q\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{(R_0 + z)(a^2 - \rho_0^2)^{1/2}} = 2\pi\frac{\mathrm{e}^{i\phi}}{\rho}[z - (l_2^2 - a^2)^{1/2}]. \tag{A5.43}$$

Yet another important integral is

$$J = \int_0^{2\pi}\int_0^a \mathrm{e}^{2i\phi_0}[z\ln(R_0 + z) - R_0]\frac{\rho_0^2 - 2a^2}{\rho_0(a^2 - \rho_0^2)^{1/2}}\mathrm{d}\rho_0 \mathrm{d}\phi_0$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left\{(l_2^2 - a^2)^{1/2}\left[\frac{a(l_2^2 + 2l_1^2 - 2a^2)}{3} - \frac{l_1^2(2l_1^2 + 3\rho^2)}{24a}\right]\right.$$

$$\left. + \frac{\rho^4}{8}\sin^{-1}(\frac{a}{l_2}) - \frac{a(\rho^2 + z^2)^{3/2}}{3} + \frac{za^3}{3}\right\}. \quad \text{(A5.44)}$$

Several partial derivatives may be computed as follows:

$$\frac{\partial J}{\partial z} = \int_0^{2\pi}\int_0^a e^{2i\phi_0}\ln(R_0 + z)\frac{\rho_0^2 - 2a^2}{\rho_0(a^2 - \rho_0^2)^{1/2}}\,d\rho_0 d\phi_0 =$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left[(a^2 - l_1^2)^{1/2}\left(l_2^2 + \frac{1}{3}l_1^2 - \frac{4}{3}a^2\right) - az(\rho^2 + z^2)^{1/2} + \frac{1}{3}a^3\right], \quad \text{(A5.45)}$$

$$\overline{\Lambda}J = -\int_0^{2\pi}\int_0^a \frac{\rho_0^2 - 2a^2}{\rho_0(a^2 - \rho_0^2)^{1/2}}\frac{\overline{q}e^{2i\phi_0}}{R_0 + z}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{i\phi}}{\rho}\left[(l_2^2 - a^2)^{1/2}\left(a - \frac{l_1^2}{2a}\right) + \frac{\rho^2}{2}\sin^{-1}(\frac{a}{l_2}) - a(\rho^2 + z^2)^{1/2}\right], \quad \text{(A5.46)}$$

$$\frac{\partial J}{\partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a e^{2i\phi_0}[z\ln(R_0 + z) - R_0]\frac{\rho_0^2 - 2a^2}{\rho_0(a^2 - \rho_0^2)^{1/2}}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left\{\frac{1}{3}(l_2^2 - a^2)^{1/2}(l_2^2 + 3l_1^2 - 4a^2) - \frac{1}{3}(\rho^2 + z^2)^{3/2} + a^2 z\right\}, \quad \text{(A5.47)}$$

$$\frac{\partial}{\partial a}(\overline{\Lambda}J) = -\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{\rho_0^2 - 2a^2}{\rho_0(a^2 - \rho_0^2)^{1/2}}\frac{\overline{q}e^{2i\phi_0}}{R_0 + z}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{i\phi}}{\rho}\left[(l_2^2 - a^2)^{1/2} - (\rho^2 + z^2)^{1/2}\right], \quad \text{(A5.48)}$$

$$\frac{\partial^2 J}{\partial a \partial z} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \mathrm{e}^{2i\phi_0}\ln(R_0+z)\,\frac{\rho_0^2-2a^2}{\rho_0(a^2-\rho_0^2)^{1/2}}\,\mathrm{d}\rho_0\mathrm{d}\phi_0$$

$$= 2\pi\frac{\mathrm{e}^{2i\phi}}{\rho^2}\left\{(a^2-l_1^2)^{1/2}\left(\frac{l_2^2}{a}-2a\right)-z(\rho^2+z^2)^{1/2}+a^2\right\}, \quad \text{(A5.49)}$$

$$\frac{\partial^2 J}{\partial z^2} = \int_0^{2\pi}\int_0^a \frac{\rho_0^2-2a^2}{R_0\rho_0(a^2-\rho_0^2)^{1/2}}\mathrm{e}^{2i\phi_0}\,\mathrm{d}\rho_0\mathrm{d}\phi_0 =$$

$$= 2\pi\frac{\mathrm{e}^{2i\phi}}{a\rho^2}\left[(l_2^2-a^2)^{1/2}(2a^2-l_1^2)-\frac{a^2(2z^2+\rho^2)}{(\rho^2+z^2)^{1/2}}\right] \quad \text{(A5.50)}$$

It is reminded that q is defined by (5.1.19). The integral in (A5.50) can be presented as a linear combination of two integrals, namely,

$$I_1 = \int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{2i\phi_0}\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{R_0(a^2-\rho_0^2)^{1/2}} = 2\pi\frac{\mathrm{e}^{2i\phi}}{a\rho^2}\left[(l_2^2-a^2)^{1/2}\left(\frac{l_2^2-a^2}{3}-\rho^2-z^2\right)+\frac{2}{3}(\rho^2+z^2)^{3/2}\right], \quad \text{(A5.51)}$$

and

$$I_2 = \int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{2i\phi_0}\,\mathrm{d}\rho_0\mathrm{d}\phi_0}{R_0\rho_0(a^2-\rho_0^2)^{1/2}} = \frac{\pi\mathrm{e}^{2i\phi}}{3a^3\rho^2}\left[\frac{3a^2(2z^2+\rho^2)+2(\rho^2+z^2)^2}{(\rho^2+z^2)^{1/2}}-2(l_2^2-a^2)^{1/2}(2a^2+l_2^2)\right]. \quad \text{(A5.52)}$$

The third basic integral is

$$N = \int_0^{2\pi}\int_0^a [z\ln(R_0+z)-R_0]\,\frac{\mathrm{e}^{i\phi_0}\,\rho_0^2\mathrm{d}\rho_0\mathrm{d}\phi_0}{(a^2-\rho_0^2)^{1/2}}$$

$$= \pi\rho\mathrm{e}^{i\phi}\left[\left(\frac{a^2}{2}+\frac{z^2}{2}-\frac{\rho^2}{8}\right)\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{2za^3}{3\rho^2}\right.$$

$$\left.+\left(\frac{5l_1^2}{8a}-\frac{a}{2}+\frac{2a^3}{3\rho^2}-\frac{al_1^2}{3\rho^2}-\frac{l_1^4}{12a\rho^2}\right)(l_2^2-a^2)^{1/2}\right]. \quad \text{(A5.53)}$$

Here follow some of the derivatives:

$$\frac{\partial N}{\partial z} = \int_0^{2\pi}\int_0^a \ln(R_0+z)\,\frac{e^{i\phi_0}\,\rho_0^2 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}}$$

$$= \pi\rho e^{i\phi}\left[z\sin^{-1}\left(\frac{a}{l_2}\right) - (a^2-l_1^2)^{1/2}\left(1-\frac{l_1^2+2a^2}{3\rho^2}\right) - \frac{2a^3}{3\rho^2}\right],$$

$$\frac{\partial^2 N}{\partial z^2} = \int_0^{2\pi}\int_0^a \frac{e^{i\phi_0}\,\rho_0^2 d\rho_0 d\phi_0}{R_0(a^2-\rho_0^2)^{1/2}} = \pi\rho e^{i\phi}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{a(l_2^2-a^2)^{1/2}}{l_2^2}\right], \quad \text{(A5.54)}$$

$$\frac{\partial^2 N}{\partial z\partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \ln(R_0+z)\,\frac{e^{i\phi_0}\,\rho_0^2 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}} = -2\pi\frac{a}{\rho}e^{i\phi}\left[a-(a^2-l_1^2)^{1/2}\right], \quad \text{(A5.55)}$$

$$\overline{\Lambda}N = -\int_0^{2\pi}\int_0^a \frac{e^{i\phi_0}\,\overline{q}\rho_0^2 d\rho_0 d\phi_0}{(R_0+z)\,(a^2-\rho_0^2)^{1/2}}$$

$$= \pi\left[\left(a^2+z^2-\frac{\rho^2}{2}\right)\sin^{-1}\left(\frac{a}{l_2}\right) - \left(a-\frac{3l_1^2}{2a}\right)(l_2^2-a^2)^{1/2}\right], \quad \text{(A5.56)}$$

$$\frac{\partial}{\partial a}\overline{\Lambda}N = -\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{e^{i\phi_0}\,\overline{q}\rho_0^2 d\rho_0 d\phi_0}{(R_0+z)\,(a^2-\rho_0^2)^{1/2}} = 2\pi a\sin^{-1}\left(\frac{a}{l_2}\right). \quad \text{(A5.57)}$$

CHAPTER 6

NEW SOLUTIONS IN FRACTURE MECHANICS

6.1. External circular crack under antisymmetric loading

Explicit expressions are derived for the field of stresses and displacements in a transversely isotropic space weakened by an external circular crack and subjected to two antisymmetrically applied concentrated forces. The presented results may be used as Green's functions for a general case of antisymmetric loading so that the complete solution can be presented in quadratures.

The external circular crack may be perceived as two elastic half-spaces connected in the plane z=0 by a circular domain which is called hereafter the crack neck. Ufliand (1967) was, probably, the first to consider the equilibrium of an *isotropic* elastic body weakened by an external circular crack and subjected to the action of two antisymmetric normal forces by an integral transform method. The same problem for the case of *transversely isotropic* body was solved in (Fabrikant, 1971d). All these solutions define the elastic field in the plane z=0 only. We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic space. One may argue that since the stresses exerted in the crack neck are known, we can substitute them into the Boussinesq point force solution (which is well known, for example, see Fabrikant, 1970) and obtain the complete solution in quadratures. Theoretically, yes, this can be done, but practically, this solution would be of little use since it would require double integration, with the integrand being singular. The computing time for this procedure would be quite significant, and its accuracy would be very doubtful. This is the main reason why, to the best of our knowledge, nobody tried so far to obtain a complete solution, even in the case of an isotropic body. On the other hand, knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems. For example, by using linear superposition of the solutions for symmetric and antisymmetric loading, we can obtain the solution to the problem of one-sided loading of a crack.

The complete solution has become possible due to the new results in potential theory given in Chapter 1. The expressions for the stresses in the crack neck are fed in the point force solution, with one important distinction: the integrals are computed in elementary functions and lead to remarkably simple and elementary expressions.

Theory. We consider a transversely isotropic elastic space weakened by an external circular crack of radius a in the plane z=0 (Fig. 6.1.1). Let two point

Fig. 6.1.1. The crack geometry

forces P be applied to the crack faces antisymmetrically in the Oz direction at the points with cylindrical coordinates $(r,\psi,0^+)$ and $(r,\psi,0^-)$. The problem, due to antisymmetric loading, can be reduced to that of a half-space $z\geq 0$, with the boundary conditions at the plane z=0

$$
\begin{aligned}
&u = 0, && \text{for } 0\leq\rho\leq a, && 0\leq\phi<2\pi;\\
&\sigma = 0, && \text{for } 0\leq\rho\leq a, && 0\leq\phi<2\pi;\\
&\sigma = P\delta(\rho-r,\phi-\psi)/\rho, && \text{for } a\leq\rho\leq\infty, && 0\leq\phi<2\pi;\\
&\tau = 0, && \text{for } a\leq\rho\leq\infty, && 0\leq\phi<2\pi.
\end{aligned}
\qquad (6.1.1)
$$

Here $\sigma=-\sigma_z$ and $\tau=-\tau_z$ as they are defined in (5.1.12). It is known (Fabrikant, 1989a) that in the case of a transversely isotropic elastic half-space subjected to a general concentrated force with the components T_x, T_y and P, the complete solution can be expressed through the three potential functions:

$$F_1 = \frac{H\gamma_1}{m_1 - 1}\left[\frac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1 + z_1)\right],$$

$$F_2 = \frac{H\gamma_2}{m_2 - 1}\left[\frac{1}{2}\gamma_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2 + z_2)\right],$$

$$F_3 = \mathrm{i}\frac{\gamma_3}{4\pi A_{44}}(\bar{\Lambda}\chi_3 - \Lambda\bar{\chi}_3). \tag{6.1.2}$$

Here (ρ_0,ϕ_0) is the point of the boundary where the concentrated force is applied;

$\chi_k(z) = \chi(z_k)$, $R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_k^2]^{1/2}$, for k=1,2,3;

$\chi(z) = T[z\ln(R_0 + z) - R_0]$, $T = T_x + \mathrm{i}T_y$,

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}. \tag{6.1.3}$$

Substitution of (2–3) in (5.1.6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\bar{T}}{R_3(R_3 + z_3)^2}\right]$$

$$+ \frac{H\gamma_2}{m_2 - 1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2} + \frac{q^2\bar{T}}{R_2(R_2 + z_2)^2}\right] + \frac{Pq}{R_2(R_2 + z_2)}\right\}$$

$$+ \frac{H\gamma_1}{m_1 - 1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1} + \frac{q^2\bar{T}}{R_1(R_1 + z_1)^2}\right] + \frac{Pq}{R_1(R_1 + z_1)}\right\}, \tag{6.1.4}$$

$$w = H\left\{\frac{1}{2}(T\bar{q} + \bar{T}q)\left[\frac{\gamma_2 m_1}{(m_1 - 1)R_1(R_1 + z_1)} + \frac{\gamma_1 m_2}{(m_2 - 1)R_2(R_2 + z_2)}\right]\right.$$

$$\left. + P\left[\frac{m_1}{(m_1 - 1)R_1} + \frac{m_2}{(m_2 - 1)R_2}\right]\right\}. \tag{6.1.5}$$

Here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \tag{6.1.6}$$

Expressions (4) and (5) simplify for the case when z=0

$$u = \frac{1}{2}G_1\frac{T}{R} + \frac{1}{2}G_2\frac{\bar{T}q^2}{R^3} - H\alpha\frac{P}{\bar{q}}, \tag{6.1.7}$$

$$w = H\alpha\Re\left(\frac{T}{q}\right) + H\frac{P}{R}. \tag{6.1.8}$$

Here $\Re$ is the sign indicating that the real part of the expression to follow is taken; H, α, G_1, and G_2 are defined by (5.1.9), and

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]^{1/2}. \tag{6.1.9}$$

Expression (7) can be used for the integral equation formulation of the problem. The governing integral equation will take the form (Fabrikant, 1971d)

$$\frac{G_1}{2}\int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0 + \frac{G_2}{2}\int_0^{2\pi}\int_0^a \frac{q^2\bar{\tau}(\rho_0,\phi_0)}{R^3}\rho_0 d\rho_0 d\phi_0 = \frac{H\alpha P}{\rho e^{-i\phi} - re^{-i\psi}} \tag{6.1.10}$$

Here τ stands for τ_z as it was defined in (5.1.10). This equation has been solved in (Fabrikant, 1971d), and its exact solution reads

$$\tau(\rho,\phi) = -\frac{2PH\alpha}{\pi^2 G_1 re^{-i\psi}(a^2-\rho^2)^{1/2}}\frac{1}{1-\bar{\zeta}}\left[1 + \left(\frac{\bar{\zeta}}{1-\bar{\zeta}}\right)^{1/2}\sin^{-1}(\bar{\zeta})^{1/2}\right]. \tag{6.1.11}$$

Here $\bar{\zeta} = \rho e^{-i\phi}/(re^{-i\psi})$. Expressions (2) can be used to obtain formulae for the potential functions in the case of a distributed loading. This will lead to computation of various integrals involving (11) and some functions of distance between points (see, for example, (4) and (5)). The simplest integral to compute is

$$I = \int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)}{R_0}\rho_0 d\rho_0 d\phi_0. \tag{6.1.12}$$

Here τ is defined by (11), and R_0 is given by (3). Let us make use of the integral representation (1.2.21)

$$\frac{1}{R_0} = \frac{2}{\pi}\int_0^{\rho_0} \lambda\left(\frac{l_1^2(x)}{\rho\rho_0}, \phi-\phi_0\right)\frac{[l_2^2(x)-x^2]^{1/2}\mathrm{d}x}{(r^2-x^2)^{1/2}[l_2^2(x)-l_1^2(x)]},$$

$$l_1(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}-[(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}+[(\rho-x)^2+z^2]^{1/2}\},$$

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}. \qquad (6.1.13)$$

and the series expansion for (11), namely,

$$\tau(\rho_0,\phi_0) = -\frac{2PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\frac{(\rho_0 e^{-i\phi_0})^n}{(re^{-i\psi})^{n+1}(a^2-\rho_0^2)^{1/2}}. \qquad (6.1.14)$$

Substitution of (13) and (14) in (12) yields, after integration with respect to ϕ_0

$$I = -\frac{8PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)(re^{-i\psi})^{n+1}}\int_0^a \rho_0\mathrm{d}\rho_0\int_0^{\rho_0}\frac{[l_2^2(x)-x^2]^{1/2}(l_1^2(x)e^{-i\phi}/\rho)^n\mathrm{d}x}{(\rho_0^2-x^2)^{1/2}(a^2-\rho_0^2)^{1/2}[l_2^2(x)-l_1^2(x)]}.$$

Changing the order of integration and consequent integration with respect to ρ_0 gives

$$I = -\frac{4PH\alpha}{\pi^{1/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)(re^{-i\psi})^{n+1}}\int_0^a\frac{[l_2^2(x)-x^2]^{1/2}(l_1^2(x)e^{-i\phi}/\rho)^n\mathrm{d}x}{l_2^2(x)-l_1^2(x)}. \qquad (6.1.15)$$

The summation in (15) can be performed, with the result

$$I = -\frac{4PH\alpha\rho e^{i\phi}}{\pi G_1}\int_0^a\left[1+\frac{l_1(x)}{[b^2-l_1^2(x)]^{1/2}}\sin^{-1}\left(\frac{l_1(x)}{b}\right)\right]\frac{[l_2^2(x)-x^2]^{1/2}\mathrm{d}x}{[l_2^2(x)-l_1^2(x)][b^2-l_1^2(x)]}. \qquad (6.1.16)$$

By introducing a new variable $y=l_1(x)$, $x=y[1+z^2/(\rho^2-y^2)]^{1/2}$, the integral (16) will take the form

$$I = -\frac{4PH\alpha\rho e^{i\phi}}{\pi G_1}\int_0^{l_1}\left[1+\frac{y}{(b^2-y^2)^{1/2}}\sin^{-1}\left(\frac{y}{b}\right)\right]\frac{dy}{(\rho^2-y^2)^{1/2}(b^2-y^2)}.$$

Throughout this section the abbreviations l_1 and l_2 denote $l_1(a)$ and $l_2(a)$ respectively. The last integral can be computed in an elementary manner, and the final result is

$$I = \frac{4PH\alpha}{\pi G_1\bar{q}}\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right]. \qquad (6.1.17)$$

Here q is defined by (6), and $b=\rho r e^{i(\phi-\psi)}$. In order to find the main potential functions (2), we need to compute the integral

$$I_1 = \int_0^{2\pi}\int_0^a \frac{\bar{q}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{R_0+z}. \qquad (6.1.18)$$

This integral can be computed from (17) by means of application of the operator $\bar{\Lambda}$to both sides of (17) and consequent integration of the result twice with respect to z. Application of $\bar{\Lambda}$ to (17) yields the following integral

$$\int_0^{2\pi}\int_0^a \frac{\bar{q}}{R_0^3}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{4PH\alpha}{\pi G_1}\frac{l_1(\rho^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)(b^2-l_1^2)}\left[1+\frac{l_1}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right]. \qquad (6.1.19)$$

Integration of both sides of (19) with respect to z results in

$$\int_0^{2\pi}\int_0^a \frac{\bar{q}}{R_0(R_0+z)}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{4PH\alpha}{\pi G_1}\left\{\frac{1}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]+\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}\right\}. \qquad (6.1.20)$$

Various formulae from Appendix 5 were used in the intermediary transformations. Yet another integration of (20) with respect to z gives

$$\int_0^{2\pi}\int_0^a \frac{\bar{q}}{R_0+z}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = \frac{4PH\alpha}{\pi G_1}\left\{\frac{z}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)$$

$$\left.+z\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}-\int_0^{l_1}\frac{x\sin^{-1}(x/b)dx}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}\right\} \qquad (6.1.21)$$

The last result allows us to define the potential functions (2) as follows:

$$F_1 = \frac{PH\gamma_1}{m_1-1}\left\{-\frac{2H\gamma_2\alpha}{\pi G_1}\left[f(z_1)+\bar{f}(z_1)\right]+\ln(R_1+z_1)\right\},$$

$$F_2 = \frac{PH\gamma_2}{m_2-1}\left\{-\frac{2H\gamma_1\alpha}{\pi G_1}\left[f(z_2)+\bar{f}(z_2)\right]+\ln(R_2+z_2)\right\},$$

$$F_3 = -\frac{i\gamma_3 PH\alpha}{\pi^2 A_{44}G_1}\left[f(z_3)-\bar{f}(z_3)\right]. \qquad (6.1.22)$$

Here the notation was introduced

$$R_k = [\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z_k^2]^{1/2}, \quad z_k = z/\gamma_k, \quad \text{for} \quad k = 1,2,3;$$

$$f(z) = \frac{z}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)$$

$$+z\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}-\int_0^{l_1}\frac{x\sin^{-1}(x/b)dx}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}. \qquad (6.1.23)$$

Now the complete solution can be obtained by substitution of (22–23) into (5.1.6) and (5.1.12). The result is

$$u = PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\Lambda\left[f(z_k)+\bar{f}(z_k)\right]+\frac{q\gamma_k}{R_k(R_k+z_k)}\right\}$$

$$+\frac{\gamma_3 PH\alpha}{\pi^2 A_{44}G_1}\Lambda\left[f(z_3)-\bar{f}(z_3)\right], \tag{6.1.24}$$

$$w = PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1\gamma_k}\frac{\partial}{\partial z_k}\left[f(z_k)+\bar{f}(z_k)\right]+\frac{1}{R_k}\right\}, \tag{6.1.25}$$

$$\sigma_1 = 2PHA_{66}\sum_{k=1}^{2}\frac{1-(1+m_k)(\gamma_3/\gamma_k)^2}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\frac{\partial^2}{\partial z_k^2}\left[f(z_k)+\bar{f}(z_k)\right]-\frac{z}{R_k^3}\right\}, \tag{6.1.26}$$

$$\sigma_2 = 2PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\Lambda^2\left[f(z_k)+\bar{f}(z_k)\right]-\frac{\gamma_k q^2(2R_k+z_k)}{R_k^3(R_k+z_k)^2}\right\}$$

$$+\frac{2PH\alpha}{\pi^2 G_1\gamma_3}\Lambda^2\left[f(z_3)-\bar{f}(z_3)\right], \tag{6.1.27}$$

$$\sigma_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^k\left\{\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\frac{\partial^2}{\partial z_k^2}\left[f(z_k)+\bar{f}(z_k)\right]+\frac{z}{R_k^3}\right\}, \tag{6.1.28}$$

$$\tau_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^k\left\{\frac{2H\alpha\gamma_1\gamma_2}{\pi\gamma_k G_1}\Lambda\frac{\partial}{\partial z}\left[f(z_k)+\bar{f}(z_k)\right]+\frac{q}{R_k^3}\right\}$$

$$+\frac{PH\alpha}{\pi^2 G_1}\Lambda\frac{\partial}{\partial z_3}\left[f(z_3)-\bar{f}(z_3)\right]. \tag{6.1.29}$$

Here are the explicit expressions for various derivatives of f which will be needed

$$\frac{\partial f}{\partial z} = \frac{1}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]+\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\,dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}, \tag{6.1.30}$$

$$\Lambda f = \frac{1}{\bar{q}}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{(\rho^2 - l_1^2)^{1/2}}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \tag{6.1.31}$$

$$\Lambda\bar{f} = \frac{l_1 e^{i\phi}(\rho^2 - l_1^2)^{1/2}}{\rho(\bar{b}^2 - l_1^2)} - \frac{z\rho_0 e^{i\phi_0}}{\bar{b}^2 - a^2}\left\{\frac{1}{(\bar{b}^2 - a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2 - a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left. - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{(\bar{b}^2 - a^2)^{1/2}}\right)\right] + \frac{a}{\bar{b}^2} - \frac{(a^2 - l_1^2)^{1/2}}{\bar{b}^2 - l_1^2}\right\} + \frac{ze^{i\phi}}{\rho}\int_0^{l_1}\frac{x^4\sin^{-1}(x/\bar{b})\,dx}{[(a^2 - x^2)(\bar{b}^2 - x^2)]^{3/2}}, \tag{6.1.32}$$

$$\frac{\partial^2 f}{\partial z^2} = -\frac{l_1(\rho^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)(b^2 - l_1^2)}\left[1 + \frac{l_1}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \tag{6.1.33}$$

$$\frac{\partial}{\partial z}\Lambda f = \frac{\rho e^{i\phi}(a^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)(b^2 - l_1^2)}\left[1 + \frac{l_1}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \tag{6.1.34}$$

$$\frac{\partial}{\partial z}\Lambda\bar{f} = \frac{(a^2 - l_1^2)^{1/2}}{\bar{b}^2 - l_1^2}\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{\rho_0 e^{i\phi_0}}{\bar{b}^2 - a^2}\right] - \frac{l_1^3 e^{i\phi}(\rho^2 - l_1^2)\sin^{-1}(l_1/b)}{\rho(a^2 - l_1^2)^{1/2}(\bar{b}^2 - l_1^2)^{3/2}(l_2^2 - l_1^2)}$$

$$- \frac{\rho_0 e^{i\phi_0} a}{\bar{b}^2(\bar{b}^2 - a^2)} - \frac{\rho_0 e^{i\phi_0}}{(\bar{b}^2 - a^2)^{3/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2 - a^2)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{(\bar{b}^2 - a^2)^{1/2}}\right)\right]$$

$$+ \frac{e^{i\phi}}{\rho}\int_0^{l_1}\frac{x^4\sin^{-1}(x/b)\,dx}{[(a^2 - x^2)(b^2 - x^2)]^{3/2}}, \tag{6.1.35}$$

$$\Lambda^2 f = -\frac{2}{\bar{q}^2}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{(\rho^2 - l_1^2)^{1/2}}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right] - \frac{e^{i\phi}(\rho^2 - l_1^2)^{1/2}}{\bar{q}(l_2^2 - l_1^2)}\left\{\frac{l_1}{\rho}\right.$$

$$\left. + \frac{\rho(a^2 - l_1^2)}{l_1(b^2 - l_1^2)} + \frac{\sin^{-1}\left(\frac{l_1}{b}\right)}{(b^2 - l_1^2)^{1/2}}\left[\frac{l_2^2}{\rho} + \frac{\rho(a^2 - l_1^2)}{b^2 - l_1^2}\right]\right\}, \tag{6.1.36}$$

$$\Lambda^2\bar{f} = \frac{z\rho_0^2 e^{2i\phi_0}}{(\bar{b}^2 - a^2)^2}\left\{\frac{3}{(\bar{b}^2 - a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2 - a^2)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{(\bar{b}^2 - a^2)^{1/2}}\right)\right] + \frac{a(5\bar{b}^2 - 2a^2)}{\bar{b}^4}\right.$$

$$-\frac{(a^2-l_1^2)^{1/2}(5\bar{b}^2-2a^2-3l_1^2)}{(\bar{b}^2-l_1^2)^2}\Bigg\}-\frac{2\rho_0 e^{i(\phi+\phi_0)}(\rho^2-l_1^2)^{1/2}}{(\bar{b}^2-l_1^2)^2}\left[\frac{l_1}{\rho}+\frac{\rho(a^2-l_1^2)}{l_1(l_2^2-l_1^2)}\right]$$

$$+\frac{e^{2i\phi}l_1(\rho^2-l_1^2)^{1/2}}{(\bar{b}^2-l_1^2)(l_2^2-l_1^2)}\left[\frac{2l_1^2-\rho^2}{\rho^2}+\frac{2(a^2-l_1^2)}{\bar{b}^2-l_1^2}-\frac{l_1^3(\rho^2-l_1^2)\sin^{-1}(l_1/\bar{b})}{\rho^2(a^2-l_1^2)[\bar{b}^2-l_1^2]^{1/2}}\right]$$

$$+\frac{3ze^{2i\phi}}{\rho^2}\int_0^{l_1}\frac{x^6\sin^{-1}(x/\bar{b})\mathrm{d}x}{(a^2-x^2)^{5/2}(\bar{b}^2-x^2)^{3/2}}. \tag{6.1.37}$$

Formulae (24–37) represent the main new results of this section. One can notice that some of the derivatives still contain uncomputed integrals, but the main advantage is that those integrals are single, rather than double, and that their integrands are non-singular which makes them easy to compute by any standard subroutine.

The main results are valid for isotropic bodies as well, provided that we substitute the elastic constants and compute the limits according to (5.1.14). These limits may be computed by using the L'Hôpital rule. The following scheme should be used:

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{f(z_1)}{m_1-1}+\frac{f(z_2)}{m_2-1}\right]=-f(z)-\frac{z}{2(1-\nu)}f'(z), \tag{6.1.38}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{m_1-1}+\frac{m_2 f(z_2)}{m_2-1}\right]=f(z)-\frac{z}{2(1-\nu)}f'(z), \tag{6.1.39}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{[1-(1+m_1)(\gamma_3/\gamma_1)^2]f(z_1)}{m_1-1}+\frac{[1-(1+m_2)(\gamma_3/\gamma_2)^2]f(z_2)}{m_2-1}\right]$$

$$=\frac{2(1+\nu)f(z)+zf'(z)}{2(1-\nu)}, \tag{6.1.40}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1 f(z_1)}{m_1-1}+\frac{\gamma_2 f(z_2)}{m_2-1}\right]=-\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \tag{6.1.41}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{\gamma_1(m_1-1)}+\frac{m_2 f(z_2)}{\gamma_2(m_2-1)}\right]=\frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)}. \tag{6.1.42}$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1=1, \quad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial\gamma_1}\right]=2(1-\nu), \tag{6.1.43}$$

and the symbol ($'$) indicates differentiation with respect to z. The field of displacements in the case of isotropy will take the form

$$u=\frac{1+\nu}{2\pi E}P\left\{\frac{zq}{R_0^3}-\frac{(1-2\nu)q}{R_0(R_0+z)}+\frac{2}{\pi}(1-2\nu)\Lambda f(z)-\frac{2\nu(1-2\nu)}{\pi(2-\nu)}\Lambda\bar{f}(z)\right.$$

$$\left.+\frac{1-2\nu}{\pi(2-\nu)}z\frac{\partial}{\partial z}\Lambda\left[f(z)+\bar{f}(z)\right]\right\}, \tag{6.1.44}$$

$$w=\frac{1+\nu}{2\pi E}P\left\{\frac{1-2\nu}{\pi(2-\nu)}\left[-(1-2\nu)[f'(z)+\bar{f}'(z)]+z[f''(z)+\bar{f}''(z)]\right]+\frac{2(1-\nu)}{R_0}+\frac{z^2}{R_0^3}\right\}. \tag{6.1.45}$$

The derivation of the field of stresses for the case of isotropy is left to the reader.

It is of interest to investigate the influence of crack neck on the field of displacements. This can be done by comparison of (44–45) with the case of an elastic half-space subjected to a normal concentrated load P which is given by the last two terms in (44–45). As we can see, the most difference will be achieved in the case of Poisson coefficient $\nu=0$, while in the other extreme, namely, $\nu=1/2$, both solutions coincide. The computations were made for the case $\nu=0$, $a=2$, $r=3$, $\psi=0$, $\phi=0$. The value of $u^*=(u/a)(2\pi E)/[P(1+\nu)]$ versus ρ/a for $z/a=0.0, 0.5, 1.0$ is plotted in Fig. 6.1.2. The negative value of ρ is understood as its value for $\phi=\pi$. A similar value of $w^*=(w/a)(2\pi E)/[P(1+\nu)]$ is plotted in Fig. 6.1.2. In both figures, the solid line curves correspond to formulae (44) and (45) respectively, while the broken line curves describe the field in an elastic half-space subjected to a normal load only. The results show that the field of normal displacements is practically unaffected even in this extreme case, while the field of tangential displacements differs significantly in the vicinity of the applied force and the crack neck. All the broken line curves

Fig. 6.1.2. The field of tangential displacements

Fig. 6.1.3. The field of normal displacements

in Fig. 6.1.2 go above the relevant solid line curves. A similar picture is observed in Fig. 6.1.3 for positive ρ, and it becomes reverse for negative ρ.

6.2. Penny-shaped crack under antisymmetric loading

Explicit expressions are derived for the field of stresses and displacements in a transversely isotropic space weakened by a penny-shaped crack and subjected to two antisymmetrically applied normal concentrated forces. The presented results may be used as Green's functions for a general case of antisymmetric loading so that the complete solution can be presented in quadratures. Several specific applications to fracture mechanics are considered.

We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic space. Though some axisymmetrical problems were considered before, we are unaware of any solution to the problem of a penny-shaped crack subjected to two antisymmetric normal forces applied at a *general* point. Knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems. For example, by using linear superposition of the solutions for symmetric and antisymmetric loading, we can obtain the solution to the problem of one-sided loading of a crack. Further application of the reciprocal theorem leads to the solution of the interaction problem between an external force and the crack.

Formulation of the antisymmetric circular crack problem. We consider a transversely isotropic elastic space weakened by a penny-shaped crack of radius a in the plane z=0 (Fig. 6.2.1). Let two point forces P be applied to the crack faces antisymmetrically in the Oz direction at the points with cylindrical coordinates $(r,\psi,0^+)$ and $(r,\psi,0^-)$. The problem, due to antisymmetric loading, can be reduced to that of a half-space $z\geq 0$, with the boundary conditions at the plane z=0

$$
\begin{aligned}
&u = 0, && \text{for } a\leq\rho\leq\infty, && 0\leq\phi<2\pi;\\
&\sigma = 0, && \text{for } a\leq\rho\leq\infty, && 0\leq\phi<2\pi;\\
&\sigma = P\delta(\rho-r,\phi-\psi)/\rho, && \text{for } 0\leq\rho\leq a, && 0\leq\phi<2\pi;\\
&\tau = 0, && \text{for } 0\leq\rho\leq a, && 0\leq\phi<2\pi.
\end{aligned}
\tag{6.2.1}
$$

It is known (Fabrikant, 1989a) that in the case of a transversely isotropic elastic half-space subjected to a general concentrated force with the components T_x, T_y and P, the complete solution can be expressed through the three potential functions:

$$
F_1 = \frac{H\gamma_1}{m_1 - 1}\left[\frac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1 + z_1)\right],
$$

Fig. 6.2.1. Loading of a penny-shaped crack

$$F_2 = \frac{H\gamma_2}{m_2 - 1}\left[\frac{1}{2}\gamma_1(\overline{\Lambda}\chi_2 + \Lambda\overline{\chi}_2) + P\ln(R_2 + z_2)\right],$$

$$F_3 = \mathrm{i}\,\frac{\gamma_3}{4\pi A_{44}}(\overline{\Lambda}\chi_3 - \Lambda\overline{\chi}_3). \qquad (6.2.2)$$

Here (ρ_0,ϕ_0) is the point of the boundary where the concentrated force is applied;

$\chi_k(z) = \chi(z_k)$, $R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_k^2]^{1/2}$, for k=1,2,3;

$\chi(z) = T[z\ln(R_0 + z) - R_0]$, $T = T_x + \mathrm{i}T_y$,

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}. \qquad (6.2.3)$$

Substitution of (2–3) in (5.1.6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\overline{T}}{R_3(R_3 + z_3)^2}\right]$$

$$+\frac{H\gamma_2}{m_2-1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2}+\frac{q^2\overline{T}}{R_2(R_2+z_2)^2}\right]+\frac{Pq}{R_2(R_2+z_2)}\right\}$$

$$+\frac{H\gamma_1}{m_1-1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1}+\frac{q^2\overline{T}}{R_1(R_1+z_1)^2}\right]+\frac{Pq}{R_1(R_1+z_1)}\right\}, \qquad (6.2.4)$$

$$w=H\left\{\frac{1}{2}(T\overline{q}+\overline{T}q)\left[\frac{\gamma_2 m_1}{(m_1-1)R_1(R_1+z_1)}+\frac{\gamma_1 m_2}{(m_2-1)R_2(R_2+z_2)}\right]\right.$$

$$\left.+P\left[\frac{m_1}{(m_1-1)R_1}+\frac{m_2}{(m_2-1)R_2}\right]\right\}. \qquad (6.2.5)$$

Here

$$q=\rho e^{i\phi}-\rho_0 e^{i\phi_0}. \qquad (6.2.6)$$

Expressions (4) and (5) simplify for the case when z=0

$$u=\frac{1}{2}G_1\frac{T}{R}+\frac{1}{2}G_2\frac{\overline{T}q^2}{R^3}-H\alpha\frac{P}{\overline{q}}, \qquad (6.2.7)$$

$$w=H\alpha\Re\left(\frac{T}{q}\right)+H\frac{P}{R}. \qquad (6.2.8)$$

Here $\Re$ is the real part sign; H, α, G_1, and G_2 are defined by (5.1.9), and

$$R=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}. \qquad (6.2.9)$$

The complete solution. Expression (7) can be used for the integral equation formulation of the problem. The governing integral equation will take the form (Fabrikant, 1989a)

$$\frac{G_1}{2}\int_0^{2\pi}\int_a^{\infty}\frac{\tau(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0+\frac{G_2}{2}\int_0^{2\pi}\int_a^{\infty}\frac{q^2\overline{\tau}(\rho_0,\phi_0)}{R^3}\rho_0 d\rho_0 d\phi_0=\frac{H\alpha P}{\rho e^{-i\phi}-re^{-i\psi}} \qquad (6.2.10)$$

Here τ stands for $-\tau_z$ as it was defined in (5.1.10). A general solution to this equation can be found in (Fabrikant, 1989a), and its exact solution in this case is elementary. It reads

$$\tau(\rho,\phi) = \frac{PH\alpha e^{i\phi}}{\pi G_1 \rho(\rho^2 - a^2)^{1/2}}\left[\frac{G_2}{G_1 - G_2} + \frac{1}{(1-\bar{\zeta})^{3/2}}\right]. \tag{6.2.11}$$

Here $\bar{\zeta} = re^{-i\psi}/(\rho e^{-i\phi})$. Expressions (2) can be used to obtain formulae for the potential functions in the case of a distributed loading. This will lead to computation of various integrals involving (11) and some functions of distance between points (see, for example, (4) and (5)). The simplest integral to compute is

$$\int_0^{2\pi}\int_a^{\infty} \frac{\tau(\rho_0,\phi_0)}{R_0}\rho_0 d\rho_0 d\phi_0. \tag{6.2.12}$$

Here τ is defined by (11), and R_0 is given by (3). We need to compute the following integral

$$I = \int_0^{2\pi}\int_a^{\infty} \frac{e^{i\phi_0} d\rho_0 d\phi_0}{R_0(\rho_0^2 - a^2)^{1/2}[1 - re^{-i\psi}/(\rho_0 e^{-i\phi_0})]^{3/2}} \tag{6.2.13}$$

Let us make use of the integral representation (see Example 13, Chapter 1)

$$\frac{1}{R_0} = \frac{2}{\pi}\int_{\rho_0}^{\infty} \lambda\left(\frac{\rho\rho_0}{l_2^2(x)}, \phi-\phi_0\right)\frac{[x^2 - l_1^2(x)]^{1/2} dx}{(x^2-\rho_0^2)^{1/2}[l_2^2(x) - l_1^2(x)]}, \tag{6.2.14}$$

where

$$l_1(x) = \frac{1}{2}\{[(\rho + x)^2 + z^2]^{1/2} - [(\rho - x)^2 + z^2]^{1/2}\},$$

$$l_2(x) = \frac{1}{2}\{[(\rho + x)^2 + z^2]^{1/2} + [(\rho - x)^2 + z^2]^{1/2}\}, \tag{6.2.15}$$

$$\lambda(k,\psi) = \frac{1-k^2}{1 + k^2 - 2k\cos\psi}, \tag{6.2.16}$$

and the series expansion

$$(1-\zeta)^{-3/2} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \zeta^n. \qquad (6.2.17)$$

Substitution of (14) and (17) in (13) yields, after integration with respect to ϕ_0

$$I = \frac{8e^{i\psi}}{r\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \int_a^{\infty} \frac{\rho_0 d\rho_0}{(\rho_0^2 - a^2)^{1/2}} \int_{\rho_0}^{\infty} \left(\frac{\rho r e^{i(\phi-\psi)}}{l_2^2(x)} \right)^{n+1} \frac{[x^2 - l_1^2(x)]^{1/2} dx}{[l_2^2(x) - l_1^2(x)](x^2 - \rho_0^2)^{1/2}}. \qquad (6.2.18)$$

Changing the order of integration in (18) and consequent integration with respect to ρ_0 and summation gives

$$I = 2\pi\rho e^{i\phi} \int_a^{\infty} \frac{l_2(x) dl_2(x)}{[l_2^2(x) - \rho^2]^{1/2} [l_2^2(x) - \rho r e^{i(\phi-\psi)}]^{3/2}} \qquad (6.2.19)$$

Here we used (17) and the substitution

$$\frac{dl_2(x)}{dx} = \frac{x(l_2^2(x) - \rho^2)}{l_2(x)[l_2^2(x) - l_1^2(x)]} \qquad (6.2.20)$$

The last integral (19) can be computed in elementary manner, and the final result is

$$I = \frac{2\pi}{\rho e^{-i\phi} - r e^{-i\psi}} \left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - b^2)^{1/2}} \right]. \qquad (6.2.21)$$

Here $b^2 = \rho r e^{i(\phi-\psi)}$. Throughout this section the abbreviations l_1 and l_2 denote $l_1(a)$ and $l_2(a)$ respectively. The last result (21) allows us to compute

$$\int_0^{2\pi} \int_a^{\infty} \frac{\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{R_0} = \frac{2PH\alpha}{G_1} \left\{ \frac{1}{\rho e^{-i\phi} - r e^{-i\psi}} \left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - b^2)^{1/2}} \right] \right.$$

$$\left. + \frac{G_2 e^{i\phi}}{(G_1 - G_2)\rho} \left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{l_2} \right] \right\}. \qquad (6.2.22)$$

In order to find the main potential functions (2), we need to compute the integral

$$I_1 = \int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{R_0+z}. \tag{6.2.23}$$

This integral can be computed from (22) by means of application of the operator $\bar{\Lambda}$ to both sides of (22) and consequent integration of the result twice with respect to z. Application of $\bar{\Lambda}$ to (22) yields the following integral

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}}{R_0^3}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{2PH\alpha}{G_1}\left[\frac{l_2^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}} + \frac{G_2}{G_1-G_2}\frac{(l_2^2-\rho^2)^{1/2}}{l_2(l_2^2-l_1^2)}\right]. \tag{6.2.24}$$

Integration of both sides of (24) with respect to z results in

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}}{R_0(R_0+z)}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{2PH\alpha}{G_1}\left\{\frac{1}{a^2-b^2}\left[aE\left(\sin^{-1}(\frac{a}{l_2}),\frac{b}{a}\right)\right.\right.$$
$$\left.\left.-\frac{b^2(l_2^2-a^2)^{1/2}}{l_2(l_2^2-b^2)^{1/2}}\right] + \frac{G_2}{(G_1-G_2)a}\sin^{-1}(\frac{a}{l_2})\right\}. \tag{6.2.25}$$

Here $E(\cdot,\cdot)$ stands for the incomplete elliptic integral of the second kind. Various formulae from Appendix 5 were used in the intermediary transformations. Yet another integration of (25) with respect to z gives

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}}{R_0+z}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{2PH\alpha}{G_1}\left[f(z) + \frac{G_2}{G_1-G_2}f_0(z)\right], \tag{6.2.26}$$

where

$$f(z) = -\frac{z}{a^2-b^2}\left[aE\left(\sin^{-1}(\frac{a}{l_2}),\frac{b}{a}\right) - \frac{b^2(l_2^2-a^2)^{1/2}}{l_2(l_2^2-b^2)^{1/2}}\right]$$
$$+\frac{(l_2^2-\rho^2)^{1/2}}{(l_2^2-b^2)^{1/2}} - \ln[(l_2^2-b^2)^{1/2} + (l_2^2-\rho^2)^{1/2}], \tag{6.2.27}$$

$$f_0(z) = -\frac{z}{a}\sin^{-1}(\frac{a}{l_2}) + \frac{(a^2-l_1^2)^{1/2}}{a} - \ln[l_2 + (l_2^2-\rho^2)^{1/2}]. \tag{6.2.28}$$

Strictly speaking, the integral in (26) is divergent, so that the right hand side in (26) represents the finite part of the divergent integral. This result was obtained by using the following convergent integral

$$\int_0^{2\pi}\int_a^{\infty}\frac{\bar{q}e^{i\phi_0}\,d\rho_0\,d\phi_0}{(R_0+z)(\rho_0^2-a^2)^{1/2}}\left[\left(1-\frac{re^{-i\psi}}{\rho_0 e^{-i\phi_0}}\right)^{-3/2}-1\right]=2\pi[-f(z)+f_0(z)]. \tag{6.2.29}$$

We notice that in the limiting case of r=0 function f coincides with f_0. Formulae (26-28) allow us to define the potential functions (2) as follows:

$$F_1=\frac{PH}{m_1-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Re\{f(z_1)\}+\frac{G_2}{G_1}f_0(z_1)\right]+\gamma_1\ln(D_1+z_1)\right\},$$

$$F_2=\frac{PH}{m_2-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Re\{f(z_2)\}+\frac{G_2}{G_1}f_0(z_2)\right]+\gamma_2\ln(D_2+z_2)\right\},$$

$$F_3=-\frac{PH\alpha\gamma_3}{\pi A_{44}G_1}\Im\{f(z_3)\}. \tag{6.2.30}$$

Here $f(\cdot)$ and $f_0(\cdot)$ are defined by (27) and (28), and the following notation was introduced:

$$D_k=[\rho^2+r^2-2\rho r\cos(\phi-\psi)+z_k^2]^{1/2},\quad z_k=z/\gamma_k,\quad \text{for } k=1,2,3; \tag{6.2.31}$$

Now the complete solution can be obtained by substitution of (30) into (5.1.6) and (5.1.12). The result is

$$u=PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Lambda\Re\{f(z_k)\}+\frac{G_2}{G_1}\Lambda f_0(z_k)\right]+\frac{(\rho e^{i\phi}-re^{i\psi})\gamma_k}{D_k(D_k+z_k)}\right\}$$

$$-\frac{i\gamma_3 PH\alpha}{\pi A_{44}G_1}\Lambda\Im\{f(z_3)\}, \tag{6.2.32}$$

$$w=PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1-\frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\}-\frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right]+\frac{1}{D_k}\right\}, \tag{6.2.33}$$

$$\sigma_1 = 2PHA_{66}\sum_{k=1}^{2}\frac{1-(1+m_k)(\gamma_3/\gamma_k)^2}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\frac{\partial^2}{\partial z_k^2}\Re\{f(z_k)\} + \frac{G_2}{G_1}\frac{(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}\right] - \frac{z}{D_k^3}\right\}, \tag{6.2.34}$$

$$\sigma_2 = 2PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Lambda^2\Re\{f(z_k)\} + \frac{G_2}{G_1}\Lambda^2 f_0(z_k)\right]\right.$$

$$\left. - \frac{\gamma_k(\rho e^{i\phi} - re^{i\psi})^2(2D_k+z_k)}{D_k^3(D_k+z_k)^2}\right\} - \frac{2PH\alpha}{\pi G_1\gamma_3}\Lambda^2\Im\{f(z_3)\}, \tag{6.2.35}$$

$$\sigma_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\frac{l_{2k}^2[l_{2k}^2-\rho^2]^{1/2}}{l_{2k}^2-l_{1k}^2}\Re\{(l_{2k}^2-b^2)^{-3/2}\}\right.\right.$$

$$\left.\left. + \frac{G_2}{G_1}\frac{(a^2-l_{1k}^2)^{1/2}}{a(l_{2k}^2-l_{1k}^2)}\right] - \frac{z}{D_k^3}\right\}, \tag{6.2.36}$$

$$\tau_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1-\frac{G_2}{G_1}\right)\Lambda\frac{\partial}{\partial z_k}\Re\{f(z_k)\} + \frac{G_2}{G_1}\Lambda\frac{\partial}{\partial z_k}f_0(z_k)\right]\right.$$

$$\left. - \frac{\rho e^{i\phi} - re^{i\psi}}{D_k^3}\right\} - \frac{PH\alpha}{\pi G_1}\Lambda\frac{\partial}{\partial z_3}\Im\{f(z_3)\}. \tag{6.2.37}$$

We recall that throughout this section the notations $\Re$ and $\Im$ stand for the real and imaginary part respectively. Here are the explicit expressions for various derivatives of f which will be needed

$$\frac{\partial f(z)}{\partial z} = -\frac{1}{a^2-b^2}\left[aE\left(\sin^{-1}\left(\frac{a}{l_2}\right), \frac{b}{a}\right) - \frac{b^2(l_2^2-a^2)^{1/2}}{l_2(l_2^2-b^2)^{1/2}}\right], \tag{6.2.38}$$

$$\Lambda f(z) = -\frac{1}{\rho e^{-i\phi} - re^{-i\psi}}\left[1 - \frac{(l_2^2-\rho^2)^{1/2}}{(l_2^2-b^2)^{1/2}}\right], \tag{6.2.39}$$

$$\Lambda\bar{f}(z)=\frac{e^{i\phi}}{\rho}\left[\frac{l_2^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-\overline{b^2})^{3/2}}-1\right]+\frac{z}{a^2-\overline{b^2}}\left\{\frac{ae^{i\phi}}{\rho}\left[F\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)-\frac{a^2+\overline{b^2}}{a^2-\overline{b^2}}E\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)\right]\right.$$

$$\left.+\frac{re^{i\psi}(l_2^2-a^2)^{1/2}}{l_2(l_2^2-\overline{b^2})^{1/2}}\left[\frac{2a^2}{a^2-\overline{b^2}}+\frac{\overline{b^2}}{l_2^2-\overline{b^2}}\right]\right\}, \tag{6.2.40}$$

$$\frac{\partial^2 f(z)}{\partial z^2}=\frac{l_2^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}} \tag{6.2.41}$$

$$\frac{\partial}{\partial z}\Lambda f(z)=\frac{l_2\rho e^{i\phi}(l_2^2-a^2)^{1/2}}{(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}}, \tag{6.2.42}$$

$$\frac{\partial}{\partial z}\Lambda\bar{f}(z)=\frac{l_2(l_2^2-a^2)^{1/2}}{(l_2^2-\overline{b^2})^{3/2}}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}-\frac{re^{i\psi}}{a^2-\overline{b^2}}\right]+\frac{ae^{i\phi}}{\rho(a^2-\overline{b^2})}\left\{F\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)\right.$$

$$\left.+\frac{a^2+\overline{b^2}}{a^2-\overline{b^2}}\left[\frac{\overline{b^2}(l_2^2-a^2)^{1/2}}{al_2(l_2^2-\overline{b^2})^{1/2}}-E\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)\right]\right\}, \tag{6.2.43}$$

$$\Lambda^2 f(z)=\frac{2}{(\rho e^{-i\phi}-re^{-i\psi})^2}\left[1-\frac{(l_2^2-\rho^2)^{1/2}}{(l_2^2-b^2)^{1/2}}\right]+\frac{\rho e^{i\phi}(l_2^2-\rho^2)^{1/2}(a^2b^2-l_2^4)}{(\rho e^{-i\phi}-re^{-i\psi})l_2^2(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}}. \tag{6.2.44}$$

The necessary derivatives of f_0 are:

$$\Lambda f_0(z)=-\frac{e^{i\phi}}{\rho}\left[1-\frac{(a^2-l_1^2)^{1/2}}{a}\right], \tag{6.2.45}$$

$$\frac{\partial}{\partial z}f_0(z)=-\frac{1}{a}\sin^{-1}(\frac{a}{l_2}), \tag{6.2.46}$$

$$\frac{\partial}{\partial z}\Lambda f_0(z)=\frac{\rho e^{i\phi}(l_2^2-a^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}, \tag{6.2.47}$$

$$\frac{\partial^2}{\partial z^2}f_0(z)=\frac{(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}, \tag{6.2.48}$$

$$\Lambda^2 f_0(z) = \frac{2e^{2i\phi}}{\rho^2}\left[1 - \frac{(a^2-l_1^2)^{1/2}}{a}\right] - \frac{e^{2i\phi}(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}. \tag{6.2.49}$$

Again, we notice that all the derivatives of f in the limiting case of r=0 coincide with those of f_0. Formulae (31–38) represent the main new results of this section.

One-sided loading of a penny-shaped crack. Consider the case when a concentrated normal force P is applied to the positive side of the crack at the point $(r, \psi, 0^+)$, while the other side is stress-free. The complete solution to this problem can be obtained by the superposition of the symmetric loading solution (see Fabrikant, 1989a) and the antisymmetric one presented here (32–37).

$$u = \frac{HP}{\pi}\left[\frac{\gamma_1}{m_1-1} f_1(z_1) + \frac{\gamma_2}{m_2-1} f_1(z_2)\right] + \frac{1}{2}PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Lambda\Re\{f(z_k)\}\right.\right.$$

$$\left.\left. + \frac{G_2}{G_1}\Lambda f_0(z_k)\right] + \frac{(\rho e^{i\phi} - re^{i\psi})\gamma_k}{D_k(D_k+z_k)}\right\} - \frac{i\gamma_3 PH\alpha}{2\pi A_{44}G_1}\Lambda\Im\{f(z_3)\}, \tag{6.2.50}$$

$$w = \frac{HP}{\pi}\left[\frac{m_1}{m_1-1} f_2(z_1) + \frac{m_2}{m_2-1} f_2(z_2)\right] + \frac{1}{2}PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1-\frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\}\right.\right.$$

$$\left.\left. - \frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right] + \frac{1}{D_k}\right\}, \tag{6.2.51}$$

$$\sigma_1 = \frac{P}{\pi^2(\gamma_1-\gamma_2)}\left\{\left[\frac{\gamma_1}{(m_1+1)\gamma_3^2} - \frac{1}{\gamma_1}\right]f_3(z_1) - \left[\frac{\gamma_2}{(m_2+1)\gamma_3^2} - \frac{1}{\gamma_2}\right]f_3(z_2)\right\}$$

$$+ PHA_{66}\sum_{k=1}^{2}\frac{1-(1+m_k)(\gamma_3/\gamma_k)^2}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\frac{\partial^2}{\partial z_k^2}\Re\{f(z_k)\} + \frac{G_2}{G_1}\frac{(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}\right] - \frac{z}{D_k^3}\right\}, \tag{6.2.52}$$

$$\sigma_2 = \frac{2}{\pi}HA_{66}P\left[\frac{\gamma_1}{m_1-1} f_4(z_1) + \frac{\gamma_2}{m_2-1} f_4(z_2)\right] + PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Lambda^2\Re\{f(z_k)\}\right.\right.$$

$$+\frac{G_2}{G_1}\Lambda^2 f_0(z_k)\bigg]-\frac{\gamma_k(\rho e^{i\phi}-re^{i\psi})^2(2D_k+z_k)}{D_k^3(D_k+z_k)^2}\bigg\}-\frac{PH\alpha}{\pi G_1\gamma_3}\Lambda^2\Im\{f(z_3)\},\tag{6.2.53}$$

$$\sigma_z=\frac{P}{2\pi^2(\gamma_1-\gamma_2)}\Big[\gamma_1 f_3(z_1)-\gamma_2 f_3(z_2)\Big]+\frac{P}{4\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\bigg\{\alpha\bigg[\bigg(1$$

$$-\frac{G_2}{G_1}\bigg)\frac{l_{2k}^2[l_{2k}^2-\rho^2]^{1/2}}{l_{2k}^2-l_{1k}^2}\Re\{(l_{2k}^2-b^2)^{-3/2}\}+\frac{G_2}{G_1}\frac{(a^2-l_{1k}^2)^{1/2}}{a(l_{2k}^2-l_{1k}^2)}\bigg]-\frac{z}{D_k^3}\bigg\},\tag{6.2.54}$$

$$\tau_z=\frac{P}{2\pi^2(\gamma_1-\gamma_2)}\Big[f_5(z_1)-f_5(z_2)\Big]+\frac{P}{4\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\bigg\{\frac{\alpha}{\gamma_k}\bigg[\bigg(1-\frac{G_2}{G_1}\bigg)\Lambda\frac{\partial}{\partial z_k}\Re\{f(z_k)\}$$

$$+\frac{G_2}{G_1}\Lambda\frac{\partial}{\partial z_k}f_0(z_k)\bigg]-\frac{\rho e^{i\phi}-re^{i\psi}}{D_k^3}\bigg\}-\frac{PH\alpha}{2\pi G_1}\Lambda\frac{\partial}{\partial z_3}\Im\{f(z_3)\}.\tag{6.2.55}$$

where

$$f_1(z)=\frac{1}{\rho e^{-i\phi}-re^{-i\psi}}\bigg[\frac{(a^2-r^2)^{1/2}}{(a^2-\overline{b}^2)^{1/2}}\tan^{-1}\frac{(a^2-\overline{b}^2)^{1/2}}{(l_2^2-a^2)^{1/2}}-\frac{z}{D}\tan^{-1}\bigg(\frac{h}{D}\bigg)\bigg],\tag{6.2.56}$$

$$f_2(z)=\frac{1}{D}\tan^{-1}\bigg(\frac{h}{D}\bigg),\tag{6.2.57}$$

$$f_3(z)=\bigg\{-\frac{z}{D^3}\tan^{-1}\bigg(\frac{h}{D}\bigg)+\frac{h}{z(D^2+h^2)}\bigg[\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{z^2}{D^2}\bigg]\bigg\},\tag{6.2.58}$$

$$f_4(z)=\frac{(a^2-r^2)^{1/2}}{(\rho e^{-i\phi}-re^{-i\psi})(a^2-\overline{b}^2)^{1/2}}\bigg(\frac{re^{i\psi}}{a^2-\overline{b}^2}$$

$$-\frac{2}{\rho e^{-i\phi}-re^{-i\psi}}\bigg)\tan^{-1}\bigg(\frac{(a^2-\overline{b}^2)^{1/2}}{(l_2^2-a^2)^{1/2}}\bigg)+\frac{z(3D^2-z^2)}{(\rho e^{-i\phi}-re^{-i\psi})^2D^3}\tan^{-1}\bigg(\frac{h}{D}\bigg)$$

$$-\frac{(a^2-r^2)^{1/2}(l_2^2-a^2)^{1/2}re^{i\psi}}{(\rho e^{-i\phi}-re^{-i\psi})(a^2-\overline{b}^2)[l_2^2-\rho re^{i(\phi-\psi)}]}$$

$$+\frac{zh}{D^2+h^2}\left[\frac{(\rho e^{i\phi}-re^{i\psi})}{(\rho e^{-i\phi}-re^{-i\psi})D^2}-\frac{\rho^2 e^{2i\phi}}{(l_2^2-l_1^2)(l_2^2-\rho^2)}\right], \tag{6.2.59}$$

$$f_5(z)=-\left\{\frac{\rho e^{i\phi}-re^{i\psi}}{D^3}\tan^{-1}\left(\frac{h}{D}\right)+\frac{h}{D^2+h^2}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}+\frac{\rho e^{i\phi}-re^{i\psi}}{D^2}\right]\right\}. \tag{6.2.60}$$

The following notation was introduced:

$$D=[\rho^2+r^2-2\rho r\cos(\phi-\psi)+z^2]^{1/2},\quad h=(a^2-r^2)^{1/2}(a^2-l_1^2)^{1/2}/a. \tag{6.2.61}$$

As one can see from the structure of equations (50–55), the solution is given by the sum of the symmetric loading solution and the antisymmetric one. The case of normal loading of the negative crack face can be obtained as the difference of the two particular solutions.

Interaction of an external force with a penny-shaped crack. Consider a transversely isotropic space weakened in the plane $z=0$ by a penny-shaped crack of radius a. Let the crack faces be stress-free, and a concentrated force be applied in the Oz direction at the point (ρ,ϕ,z). Solution to this problem can be obtained by application of the reciprocal theorem to the general results obtained here. Let a unit normal force N be applied to the positive face of the crack at the point $(r,\psi,0^+)$. Denote w_N the normal displacements at the point (ρ,ϕ,z) due to the unit force N, and denote w_P the normal displacements at the point $(r,\psi,0^+)$ due to the external force P. Application of the reciprocal theorem immediately gives that

$$w_P=Pw_N, \tag{6.2.62}$$

which translates into

$$w(r,\psi,0^+)=\frac{PH}{\pi}\left[\frac{m_1}{m_1-1}f_2(z_1)+\frac{m_2}{m_2-1}f_2(z_2)\right]+\frac{1}{2}PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1\right.\right.\right.$$

$$\left.\left.\left.-\frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\}-\frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right]+\frac{1}{D_k}\right\}. \tag{6.2.63}$$

The normal displacements of the negative face of the crack can be found in a similar manner, with the result

$$w(r,\psi,0^-) = -\frac{PH}{\pi}\left[\frac{m_1}{m_1-1} f_2(z_1) + \frac{m_2}{m_2-1} f_2(z_2)\right] + \frac{1}{2}PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1 - \frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\} - \frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right] + \frac{1}{D_k}\right\}. \quad (6.2.64)$$

The crack opening displacement can be obtained as the difference between (63) and (64). Computation of the crack shape was made for a transversely isotropic body, with the following values assigned to the elastic constants: $A_{11}=A_{33}=2.7777$, $A_{44}=A_{66}=1$, $A_{13}=0.6944$. The dimensionless quantity $w^*=wa/(PH)$ versus ρ/a is presented in Fig. 6.2.2 for $r/a=1.5$, $\phi=0$, $\psi=0$, and $z/a=0,1,2$. Since the crack

Fig. 6.2.2. Crack shape due to an external force

opening displacement is zero for $z=0$, and it tends to zero for $z\to\infty$, there should be a location for an external force where it produces maximum crack opening.

Discussion. Some particular cases of interest are considered here as well as some applications to various problems in fracture mechanics.

The main results are valid for isotropic bodies provided that we substitute the elastic constants and compute the limits according to (5.1.13). These limits may be computed by using the L'Hôpital rule (see 6.1.38–6.1.43) for the scheme involved. The field of displacements in the case of isotropy will take the form

$$u=\frac{(1+\nu)P}{2\pi E}\left\{-\frac{(1-2\nu)}{2-\nu}\left[(2-\nu)\Lambda f(z)+\nu\Lambda[f_0(z)-\bar{f}(z)]\right.\right.$$

$$\left.\left.+z\Lambda\frac{\partial}{\partial z}\left(\Re\{f(z)\}+\frac{\nu}{2(1-\nu)}f_0(z)\right)\right]+\frac{\rho e^{i\phi}-re^{i\psi}}{D}\left[\frac{z}{D^2}-\frac{1-2\nu}{D+z}\right]\right\}, \tag{6.2.65}$$

$$w=\frac{(1+\nu)P}{2\pi E}\left\{\frac{(1-2\nu)^2}{2-\nu}\left[\Re\{f'(z)\}+\frac{\nu}{2(1-\nu)}f_0'(z)\right]\right.$$

$$\left.-\frac{1-2\nu}{2-\nu}z\left[\Re\{f''(z)+\frac{\nu}{2(1-\nu)}f_0''(z)\right]+\frac{2(1-\nu)}{D}+\frac{z^2}{D^3}\right\}. \tag{6.2.66}$$

A comparison can be made between the field of displacements due to a concentrated normal loading of a half-space with the field defined by (65) and (66). Both fields coincide for $\nu=1/2$, and their difference increases with ν decreasing. Computations were made for $\nu=0$, $r=0.5a$, and different values of z. The dimensionless parameter $u^*=2\pi Eu/[(1+\nu)P]$ versus ρ/a is presented in Fig. 6.2.3. The value of $w^*=2\pi Ew/[(1+\nu)P]$ versus ρ/a for $z/a=0,2$ is presented in Fig. 6.2.4. Solid line curves in both figures correspond to formulae (65) and (66) respectively, while the broken line curves give the results due to the concentrated loading of the half-space. The derivation of the field of stresses for the case of isotropy is left to the reader.

In the case of axial symmetry r=0, and the complete solution can be expressed in terms of elementary functions. The main potential functions will take the form

$$F_1=\frac{PH}{m_1-1}\left[\alpha f_0(z_1)+\gamma_1\ln[(\rho^2+z_1^2)^{1/2}+z_1]\right], \tag{6.2.67}$$

$$F_2=\frac{PH}{m_2-1}\left[\alpha f_0(z_2)+\gamma_2\ln[(\rho^2+z_2^2)^{1/2}+z_2]\right], \tag{6.2.68}$$

$$F_3=0. \tag{6.2.69}$$

We recall that f_0 is defined by (28). The field of displacements is

Fig. 6.2.3. Comparison of tangential displacements

Fig. 6.2.4. Comparison of normal displacements

$$u = PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\alpha\,\frac{e^{i\phi}}{\rho}\left[1-\frac{(a^2-l_{1k}^2)^{1/2}}{a}\right]+\frac{\rho e^{i\phi}\gamma_k}{(\rho^2+z_k^2)^{1/2}[(\rho^2+z_k^2)^{1/2}+z_k]}\right\}, \tag{6.2.70}$$

$$w = PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k a}\sin^{-1}\left(\frac{a}{l_{2k}}\right) + \frac{1}{(\rho^2+z_k^2)^{1/2}}\right\}. \tag{6.2.71}$$

Again, we leave the derivation of the field of stresses to the reader. It can be done easily by using the appropriate differentiation formulae from Appendix 5.

The stress intensity factors (SIF) play an important role in fracture mechanics. The solution presented can be used for computing the SIF in various particular cases. Define the three mode stress intensity factors as follows:

$$k_1 = \lim_{\rho\to a}\{(\rho-a)^{1/2}\sigma_z\}, \tag{6.2.72}$$

$$k_2 + ik_3 = \lim_{\rho\to a}\{(\rho-a)^{1/2}\tau_z e^{-i\phi}\}, \tag{6.2.73}$$

Since the normal stress in the plane z=0 vanishes in the case of antisymmetric loading, so does the mode I SIF k_1. The remaining SIF can be obtained from (11) as follows:

$$k_2(\phi) + ik_3(\phi) = \frac{PH\alpha}{\pi a\sqrt{2a}G_1}\left\{\frac{G_2}{G_1-G_2} + \frac{1}{[1-(r/a)e^{i(\phi-\psi)}]^{3/2}}\right\}. \tag{6.2.74}$$

A similar result can be obtained by using the general expression of SIF through the limiting values of displacements (Fabrikant, 1989a)

$$k_2 + ik_3 = -\frac{a}{\pi(G_1^2-G_2^2)\sqrt{2a}}\lim_{\rho\to a}\left[\frac{G_1 e^{-i\phi}u(\rho,\phi) + G_2 e^{i\phi}\overline{u}(\rho,\phi)}{(a^2-\rho^2)^{1/2}}\right]. \tag{6.2.75}$$

Here u stands for the complex tangential displacements in the plane z=0. Indeed, one can get from (32)

$$u(\rho,\phi,0) = PH\alpha(a^2-\rho^2)^{1/2}\left\{\frac{G_2 e^{i\phi}}{G_1\rho}\left[\frac{a^2}{(a^2-\overline{b^2})^{3/2}} - \frac{1}{a}\right]\right.$$

$$\left. - \frac{1}{(\rho e^{-i\phi} - re^{-i\psi})(a^2-b^2)^{1/2}}\right\}, \tag{6.2.76}$$

and the substitution of (76) in (75) leads to (74).

In the case of one-sided loading of a penny-shaped crack, the SIF are

$$k_1(\phi) = \frac{P}{2\pi^2\sqrt{2a}} \frac{(a^2 - r^2)^{1/2}}{a^2 + r^2 - 2ar\cos(\phi - \psi)}, \tag{6.2.77}$$

$$k_2(\phi) + ik_3(\phi) = \frac{PH\alpha}{2\pi a\sqrt{2a}G_1}\left\{\frac{G_2}{G_1 - G_2} + \frac{1}{[1-(r/a)e^{i(\phi-\psi)}]^{3/2}}\right\}. \tag{6.2.78}$$

Formulae (77) and (78) allow us to consider some cases of one-sided distributed loading. Let the load of intensity p per unit length be uniformly distributed along a circle of radius r. Direct integration results in

$$k_1 = \frac{p}{\pi\sqrt{2a}(a^2 - r^2)^{1/2}}, \tag{6.2.79}$$

$$k_2 = \frac{p\alpha}{2a\gamma_1\gamma_2\sqrt{2a}}, \quad k_3 = 0. \tag{6.2.80}$$

Notice that k_2 does not depend on the loading location. The case of uniform loading p distributed over an annulus $r_1 \leq r \leq r_2$ can be considered in a similar manner. The result is

$$k_1 = \frac{p[(a^2 - r_1^2)^{1/2} - (a^2 - r_2^2)^{1/2}]}{\pi\sqrt{2a}}, \tag{6.2.81}$$

$$k_2 = \frac{p\alpha[r_2^2 - r_1^2]}{4a\gamma_1\gamma_2\sqrt{2a}}. \tag{6.2.82}$$

Introduction of the total load $P = \pi p(r_2^2 - r_1^2)$ transforms (81) and (82) into

$$k_1 = \frac{P}{\pi^2\sqrt{2a}[(a^2 - r_1^2)^{1/2} + (a^2 - r_2^2)^{1/2}]}, \tag{6.2.83}$$

$$k_2 = \frac{P\alpha}{4\pi a\gamma_1\gamma_2\sqrt{2a}}.$$

Again, the value of k_2 does not depend on the location of the loading.

6.3. Annular crack under general normal loading

The flat annular crack in an elastic space under the action of a uniform pressure was considered by a number of authors starting with the early work of Smetanin (1968) and ending with the most recent work of Clements and Ang (1988) where the reader can find additional references. All these publications treat only the simplest case of uniform loading. To our knowledge, no attempt has been made to solve the general non-axisymmetric problem. Such a solution is presented here for the case of a transversely isotropic body. The method is based on the results described in Fabrikant (1989a). The problem is reduced to a set of two two-dimensional integral equations of Fredholm type with elementary kernels. The equations can be easily uncoupled and solved numerically with high accuracy.

Formulation of the problem. Consider a transversely isotropic elastic space weakened in the plane $z=0$ by a flat annular crack $b \le \rho \le a$. An arbitrary pressure $p(\rho,\phi)$ is applied to the crack faces in opposite directions. Due to the symmetry, the problem can be reduced to a mixed boundary value problem for an elastic half-space, with the following boundary conditions prescribed on the plane $z=0$:

$$w=0 \quad \text{for} \quad 0<\rho<b \quad \text{or} \quad \text{for} \quad \rho>a;$$

$$\sigma_z=-p(\rho,\phi) \quad \text{for} \quad b\le\rho\le a. \tag{6.3.1}$$

Here w denotes the normal displacement, and σ_z is the normal stress. If a normal concentrated load P is applied to the boundary of the half-space at the point $(\rho_0,\phi_0,0)$, then the normal displacements at $z=0$ can be defined as (Fabrikant, 1989a)

$$w(\rho,\phi)=H\frac{P}{R} \tag{6.3.2}$$

Here H is an elastic constant which in the case of isotropy transforms into $(1-\nu^2)/(\pi E)$, and

$$R=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{6.3.3}$$

In the case where the normal pressure σ is prescribed all over the plane $z=0$, the relevant normal displacements can be related to it in two different ways, namely, (Fabrikant, 1989b):

$$w(\rho,\phi)=4H\int_0^\rho \frac{dx}{(\rho^2-x^2)^{1/2}}\int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{1/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \tag{6.3.4}$$

$$w(\rho,\phi) = 4H\int_{\rho}^{\infty}\frac{dx}{(x^2-\rho^2)^{1/2}}\int_0^x\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{6.3.5}$$

Here the $\mathcal{L}$-operator is defined as

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi}\lambda(k,\phi-\phi_0)\,f(\phi_0)\,d\phi_0,$$

with

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}.$$

We can not use (4) and (5) directly since σ is known only on the interval $b \le \rho \le a$. Let us introduce two new unknowns σ_1 and σ_2 which are assumed to be equal to the normal pressure in the intervals $0 \le \rho < b$ and $\rho > a$ respectively. These unknowns can be found from the first boundary condition (1). In the case $\rho > a$ it is convenient to use (5) which can be rewritten as

$$0 = 4H\int_{\rho}^{\infty}\frac{dx}{(x^2-\rho^2)^{1/2}}\left\{\int_0^b\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_1(\rho_0,\phi)\right.$$

$$\left.+\int_b^a\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)p(\rho_0,\phi)+\int_a^x\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_2(\rho_0,\phi)\right\},$$

which immediately leads to

$$\int_0^b\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_1(\rho_0,\phi)+\int_b^a\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho_0}{x}\right)p(\rho_0,\phi)$$

$$+\int_a^x\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_2(\rho_0,\phi)=0 \tag{6.3.6}$$

Application of the operator

$$\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_a^{\rho}\frac{x\,dx}{(\rho^2-x^2)^{1/2}}\,\mathcal{L}(x)$$

to both sides of (6) leads to the first governing equation

$$\sigma_2(\rho,\phi)+\frac{2}{\pi}\frac{1}{(\rho^2-a^2)^{1/2}}\int_0^b\frac{(a^2-\rho_0^2)^{1/2}}{\rho^2-\rho_0^2}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma_1(\rho_0,\phi)\,\rho_0\,d\rho_0$$

$$=-\frac{2}{\pi}\frac{1}{(\rho^2-a^2)^{1/2}}\int_b^a\frac{(a^2-\rho_0^2)^{1/2}}{\rho^2-\rho_0^2}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)p(\rho_0,\phi)\,\rho_0\,d\rho_0\quad\text{for}\quad\rho>a.$$

(6.3.7)

A similar utilization of (4) for $0\le\rho\le b$ leads to yet another equation

$$\sigma_1(\rho,\phi)+\frac{2}{\pi}\frac{1}{\sqrt{b^2-\rho^2}}\int_a^{\infty}\frac{\sqrt{\rho_0^2-b^2}}{\rho_0^2-\rho^2}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma_2(\rho_0,\phi)\,\rho_0\,d\rho_0$$

$$=-\frac{2}{\pi}\frac{1}{\sqrt{b^2-\rho^2}}\int_b^a\frac{\sqrt{\rho_0^2-b^2}}{\rho_0^2-\rho^2}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right)p(\rho_0,\phi)\,\rho_0\,d\rho_0\quad\text{for}\quad 0\le\rho<b.$$

(6.3.8)

Though the kernels of integral equations (7) and (8) look different, a special transformation can make them identical. Indeed, let us substitute in (8) the new variables x and t which are related to the old ones ρ and ρ_0 as follows: $\rho=\sqrt{ab}\,t$ and $\rho_0=\sqrt{ab}/x$. The result reads

$$\sigma_1(\sqrt{ab}\,t,\phi)+\frac{2}{\pi}\frac{1}{\sqrt{k^2-t^2}}\int_0^k\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,\sigma_2(\sqrt{ab}/x,\phi)\,\frac{dx}{x^2}$$

$$=-\frac{2}{\pi}\frac{1}{\sqrt{k^2-t^2}}\int_k^{1/k}\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,p(\sqrt{ab}/x,\phi)\,\frac{dx}{x^2}\qquad(6.3.9)$$

In a similar manner, substitution of $\rho=\sqrt{ab}/t$ and $\rho_0=\sqrt{ab}\,x$ in (7) results in

$$\sigma_2(\sqrt{ab}/t,\phi)+\frac{2}{\pi}\frac{t^3}{\sqrt{k^2-t^2}}\int_0^k\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,\sigma_1(\sqrt{ab}\,x,\phi)\,x\,\mathrm{d}x$$

$$=-\frac{2}{\pi}\frac{t^3}{\sqrt{k^2-t^2}}\int_0^k\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,p(\sqrt{ab}\,x,\phi)\,x\,\mathrm{d}x. \tag{6.3.10}$$

Equations (9) and (10) can be rewritten as

$$s_1(t,\phi)+\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^k\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,s_2(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_1(t,\phi), \tag{6.3.11}$$

$$s_2(t,\phi)+\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^k\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,s_1(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_2(t,\phi). \tag{6.3.12}$$

Here the following notation has been introduced

$$s_1(t,\phi)=\sigma_1(\sqrt{ab}t,\phi),\quad s_2(t,\phi)=\sigma_2(\sqrt{ab}/t,\phi)/t^3,\quad R_{xt}^2=1+x^2t^2-2xt\cos(\phi-\phi_0), \tag{6.3.13}$$

$$g_1(t,\phi)=-\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_k^{1/k}\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,p(\sqrt{ab}/x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0$$

$$g_2(t,\phi)=-\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_k^{1/k}\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,p(\sqrt{ab}x,\phi_0)\,\frac{\mathrm{d}x}{x^2}\,\mathrm{d}\phi_0. \tag{6.3.14}$$

Equations (11) and (12) can be easily uncoupled by simple summation and subtraction

$$s_+(t,\phi)+\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^k\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,s_+(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_+(t,\phi), \tag{6.3.15}$$

$$s_-(t,\phi) - \frac{1}{\pi^2\sqrt{k^2-t^2}} \int_0^{2\pi}\int_0^k \frac{\sqrt{1-k^2x^2}}{R_{xt}^2} s_-(x,\phi_0)\, x\,\mathrm{d}x\,\mathrm{d}\phi_0 = g_-(t,\phi). \quad (6.3.16)$$

Here we denoted

$$s_\pm = s_1 \pm s_2, \quad g_\pm = g_1 \pm g_2. \quad (6.3.17)$$

Equations (15) and (16) seem to be new. They represent the main result of this section.

Examples. Equations (15) and (16) can be solved numerically for arbitrary g_1 and g_2. One should though eliminate the singularity by multiplying both sides of each equation by the term $\sqrt{k^2-t^2}$. In the case of axial symmetry, equations (15) and (16) simplify as follows:

$$s_+(t) + \frac{2}{\pi\sqrt{k^2-t^2}} \int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2} s_+(x)\, x\,\mathrm{d}x = g_+(t), \quad (6.3.18)$$

$$s_-(t) - \frac{2}{\pi\sqrt{k^2-t^2}} \int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2} s_-(x)\, x\,\mathrm{d}x = g_-(t). \quad (6.3.19)$$

We rewrite equations (18) and (19) as

$$S_+(t) + \frac{2}{\pi} \int_0^k \frac{\sqrt{1-k^2x^2}}{\sqrt{k^2-x^2}\,(1-x^2t^2)} S_+(x)\, x\,\mathrm{d}x = h_+(t), \quad (6.3.20)$$

$$S_-(t) - \frac{2}{\pi} \int_0^k \frac{\sqrt{1-k^2x^2}}{\sqrt{k^2-x^2}\,(1-x^2t^2)} S_-(x)\, x\,\mathrm{d}x = h_-(t). \quad (6.3.21)$$

The relevant notation is

$$S_\pm(t) = s_\pm(t)\sqrt{k^2-t^2}, \quad h_\pm(t) = g_\pm(t)\sqrt{k^2-t^2}. \quad (6.3.22)$$

Consider in more detail the case of a uniform loading $p = \text{const.}$ In this case formulae (14) yield

$$h_{\pm}(t) = -\frac{2}{\pi} p \left\{ \frac{\sqrt{1-k^4}}{k} - \sqrt{k^2 - t^2} \ \sin^{-1} \frac{\sqrt{1-k^4}}{\sqrt{1-k^2 t^2}} \right.$$

$$\left. \pm \frac{1}{t^2} \left[\sqrt{1-k^4} - \frac{\sqrt{k^2-t^2}}{t} \sin^{-1} \frac{t\sqrt{1-k^4}}{k\sqrt{1-k^2t^2}} \right] \right\}. \tag{6.3.23}$$

Now we give a description of the numerical method used for solving (9) and (10). Consider the following integral equation:

$$S(t) + \int_0^k \mathcal{K}(t,x)\, S(x)\, \mathrm{d}x = h(t). \tag{6.3.24}$$

Here h is a known function, $\mathcal{K}$ is the kernel, and S is the as yet unknown function. In the numerical methods used in the literature it is assumed usually that the unknown function is piecewise constant. We consider the unknown function S to be piecewise *linear*. The notation S stands for either S_+ or S_-. We divide the interval $[0, k]$ into $n-1$ equal subintervals of length $\Delta = k/(n-1)$. The points of division are called x_l, $l = 1, 2, \ldots, n$. Let $S_l = S(x_l)$, for $l = 1, 2, \ldots, n$. This implies that at the l-th subinterval the function S can be expressed as follows:

$$S(x) = S_l + (S_{l+1} - S_l)\left(\frac{x}{\Delta} - l\right), \quad \text{for} \quad x_l < x < x_{l+1}. \tag{6.3.25}$$

Substitution of (25) in (20) or (21) leads to a set of n linear algebraic equations

$$S_l + S_1 \left[\kappa_1(t_l) - \frac{\theta_1(t_l)}{\Delta} \right] + \sum_{i=2}^{n\text{-}1} S_i \left[i\kappa_i(t_l) - (i-2)\kappa_{i-1}(t_l) - \frac{\theta_i(t_l) - \theta_{i-1}(t_l)}{\Delta} \right]$$

$$+ S_n \left[\frac{\theta_{n-1}(t_l)}{\Delta} - (n-2)\kappa_{n-1}(t_l) \right] = h(t_l), \quad t_l = x_l, \quad \text{for} \quad l = 1, 2, \ldots, n. \tag{6.3.26}$$

Here

$$\kappa_i(t_l) = \int_{x_i}^{x_{i+1}} \mathcal{K}(t_l, x)\,\mathrm{d}x. \tag{6.3.27}$$

$$\theta_i(t_l) = \int_{x_i}^{x_{i+1}} \mathcal{K}(t_l, x)\,x\,\mathrm{d}x. \tag{6.3.28}$$

Since the piecewise linear function follows the real function more close than the piecewise constant one, we should expect the set of equations (26) to give a more accurate solution.

Actual computations were made with $n=301$ and the set of values (b/a): 0.04, 0.1, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99. The dimensionless quantities $S_1^*=S_1/p$ and $S_2^*=S_2/p$ are plotted in Fig. 6.3.1 and Fig. 6.3.2 respectively versus the quantity $\xi=1+300(t/k)$. This choice allowed us to plot all curves on the same base. In

Fig. 6.3.1. Annular crack under uniform normal loading (solution for S_1)

order to avoid overlapping, not all curves are actually plotted in Fig. 6.3.1 and 6.3.2. The stress intensity factors were defined as

$$K_a = \lim_{\rho\to a} \{\sigma_2(\rho)\sqrt{\rho-a}\}, \quad K_b = \lim_{\rho\to b} \{\sigma_1(\rho)\sqrt{b-\rho}\}. \tag{6.3.29}$$

Fig. 6.3.2. Annular crack under uniform normal loading (solution for S_2)

They can be simply defined in terms of the functions S_1 and S_2, namely,

$$K_a = k^2\sqrt{a/2}\, S_2(k), \quad K_b = \sqrt{a/2}\, S_1(k). \tag{6.3.30}$$

We have normalized these quantities by the stress intensity factor K_0 for a penny-shaped circular crack of radius a subjected to the same pressure p

$$K_0 = \frac{\sqrt{2a}}{\pi} p,$$

with the result

$$K_a^* = \frac{K_a}{K_0} = \frac{\pi k^2}{2p} S_2(k), \quad K_b^* = \frac{K_b}{K_0} = \frac{\pi}{2p} S_1(k). \tag{6.3.31}$$

The absolute values of the computed results are presented in the Table below

b/a	0.04	0.1	0.2	0.4	0.6	0.8	0.9	0.95	0.99
$\lvert K_b^*\rvert$	3.2245	2.0742	1.4892	1.0339	0.7646	0.5113	0.3541	0.2473	0.1064
$\lvert K_a^*\rvert$	0.9835	0.9578	0.9118	0.8055	0.6713	0.4848	0.3460	0.2450	0.1065

The results above are in excellent agreement with those of Clements and Ang

(1988) who have normalized their results relative to a straight two-dimensional crack of length $a-b$. As one should expect, the accuracy deteriorates as k approaches unity. Indeed, the results for $(b/a)=0.99$ are qualitatively incorrect since they give $K_b<K_a$. If one assumes the results of Clements and Ang as exact, then the correct values for this case are $K_b^*=-0.1112$ and $K_a^*=-0.1108$, with the error of about 5%. The accuracy of the other data is expected to be much better than that. Our results should be multiplied by the factor $2\sqrt{2}/(\pi\sqrt{1-k^2})$ in order to be compared with those of Clements and Ang.

The solution accuracy was also verified by checking the condition of equilibrium of the plane $z=0$, namely,

$$P+P_1+P_2=0. \tag{6.3.32}$$

Here

$$P=\pi p(a^2-b^2),\quad P_1=2\pi\int_0^b \sigma_1(\rho)\,\rho\,d\rho=2\pi ab\int_0^k \frac{S_1(t)\,t\,dt}{\sqrt{k^2-t^2}},$$

$$P_2=2\pi\int_a^\infty \sigma_2(\rho)\,\rho\,d\rho=2\pi ab\int_0^k \frac{S_2(t)\,dt}{\sqrt{k^2-t^2}}. \tag{6.3.33}$$

The condition (32) was satisfied with high accuracy which also deteriorates for k close to unity. For example, the discrepancy in equilibrium conditions for $(b/a)=0.99$ was about 11%. In the case of a very narrow annulus we need to use either n greater than 301 or to use the asymptotic two-dimensional solution.

The case of non-axisymmetric loading can be considered in a similar manner. For example, when the loading pressure can be described by a first harmonic, namely,

$$p(\rho,\phi)=p_1\rho\cos\phi, \tag{6.3.34}$$

the governing integral equations will take the form

$$s_+(t)+\frac{2t}{\pi\sqrt{k^2-t^2}}\int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,s_+(x)\,x^2\,dx=g_+(t), \tag{6.3.35}$$

$$s_-(t)-\frac{2t}{\pi\sqrt{k^2-t^2}}\int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,s_-(x)\,x^2\,dx=g_-(t), \tag{6.3.36}$$

where

$$g_{\pm}(t) = -\frac{2p_1 ak}{\pi t}\left\{\frac{\sqrt{1-k^4}}{\sqrt{k^2-t^2}} - \frac{1}{t}\sin^{-1}\frac{t\sqrt{1-k^4}}{k\sqrt{1-k^2t^2}} \pm \left[\frac{k\cos^{-1}(k^2)}{\sqrt{k^2-t^2}} - \cos^{-1}\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right]\right\}. \tag{6.3.37}$$

Again, singularities can be eliminated by an appropriate change of variables. The resulting equations can be written in a compact form

$$\psi_{\pm}(t) \pm \frac{2}{\pi}\int_0^k \frac{\sqrt{1-k^2x^2}}{\sqrt{k^2-x^2}(1-x^2t^2)}\,\psi_{\pm}(x)\,x^3\,\mathrm{d}x = h_{\pm}(t), \tag{6.3.38}$$

where

$$\psi_{\pm}(t) = s_{\pm}(t)\sqrt{k^2-t^2}/t, \quad h_{\pm}(t) = g_{\pm}(t)\sqrt{k^2-t^2}/t. \tag{6.3.39}$$

Note that both functions ψ and h are finite at $t=0$. We may deduce

$$h_{\pm}(0) = -\frac{1}{\pi}\left\{\frac{2(1-k^4)^{3/2}}{3k} \pm \left[\cos^{-1}(k^2) - \sqrt{1-k^4}\right]\right\}.$$

Equations (38) can be solved numerically. We note that the annular crack problem can also be solved by the method described in the section 3.7.

CHAPTER 7

SINGULAR INTEGRAL EQUATIONS

7.1. Approximate solution of singular integral equations

A method for obtaining the approximate solution of singular integral equations of the first and second kind is suggested. The solution is represented in the form of power series with undetermined coefficients multiplied by a function in which the essential features of the singularity of the solution are preserved. The method of collocations is used to determine the unknown coefficients. The examples show that the method suggested is more general and gives good results even in the case when the form of solution does not exactly preserve the essential features of singularity. The method is simpler than others which use the properties of orthogonal polynomials, and is applicable for the solution of single equations as well as systems of simultaneous equations.

Singular integral equations have numerous applications in many branches of mechanics as well as in electrostatics. Such applications include problems of plane flow around an arc, problems of rock mechanics, composite materials and layered media containing cuts, crack propagation in elastic plates, contact problems of the plane theory of elasticity with or without friction, diffusion and wave diffraction problems in media containing geometric or physical singularities and aerofoil problems, etc.

The main results concerning exact solutions of singular integral equations and methods of reducing them to a Fredholm integral equation with continuous and bounded kernel are presented by Muskhelishvili (1953). But even when the exact solution does exist, it is difficult to obtain numerical results because such a solution has the form of a singular integral which is often not suitable for numerical integration. Therefore approximate methods of solution of singular integral equations are of great importance. In 1941 Hildebrand suggested an approximate solution of singular integral equations in the form of a sum of power series with unknown coefficients and a function with prescribed singular properties. Here, the coefficients were determined by the method of least squares. This method is simple but not sufficiently accurate. In more general terms the approximate solution of singular integral equations was also discussed in Ivanov (1956) and Kalandiia (1959). Useful results were obtained by using the orthogonal polynomials properties in Karpenko (1966), Popov (1966) and Erdogan (1969). This method is accurate but is more complicated than Hildebrand's (1941) since it requires a good knowledge of special functions such as Chebyshev and Jacobi polynomials. The finite element approach (D. Q. Dang, D. H. Norrie 1979) to the solution of a singular integral equation proved to be simple but it requires many points of collocations to obtain good accuracy and it fails for example in the case when the stress intensity factor is to be calculated.

The method proposed in this section differs form that of Hildebrand on two points namely: (a) instead of summing, the solution of a singular integral equation is sought in the form of a product of a power series with unknown coefficients and a function with the prescribed singular properties, and (b) the method of collocation is used for the determination of unknown coefficients. These two important differences make the method as accurate as the one using the properties of the orthogonal polynomials and much simpler than any of the other methods available for generating numerical results.

Description of the method. We consider the singular integral equation

$$a\,f(x) + b\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{t-x} + \int_{-1}^{1}k(x,t)\,f(t)\,\mathrm{d}t = G(x). \tag{7.1.1}$$

Here a and b are constants, $k(x,t)$ is bounded, and $G(x)$ satisfies all the necessary conditions (see Muskhelishvili, 1953) in the interval $-1 \le x \le 1$.

In the case when the solution $f(x)$ at $x = \pm 1$ is unbounded but integrable, one can assume the solution of (1) to take the form

$$f(x) = \sum_{n=0}^{N}\frac{C_n x^n}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}}. \tag{7.1.2}$$

Here $\alpha = \pi^{-1}\cot^{-1}(\pi b/a)$.

Substitution of solution (2) into (1) yields after integrating

$$b\sum_{n=1}^{N} C_n \sum_{l=0}^{n-1} B_l\, x^{n-l-1} + \sum_{n=0}^{N} C_n \phi_n(x) = G(x). \tag{7.1.3}$$

Here the following integrals (Gradshtein and Ryzhik, 1965) have been employed

$$\int_{-1}^{1} \frac{x^k \,\mathrm{d}x}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}} = (-1)^k B(\tfrac{1}{2}-\alpha, \tfrac{1}{2}+\alpha)\, F(-k, \tfrac{1}{2}-\alpha; 1; 2),$$

$$\int_{-1}^{1} \frac{\mathrm{d}x}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}(x-y)} = -\frac{\pi \tan(\pi\alpha)}{(1+y)^{1/2+\alpha}(1-y)^{1/2-\alpha}},$$

where B denotes the Beta-function; and F denotes the Gauss hypergeometric function. Also

$$\phi_n(x) = \int_{-1}^{1} \frac{k(x,t)t^n \mathrm{d}t}{(1+t)^{1/2+\alpha}(1-t)^{1/2-\alpha}}, \quad B_l = \frac{\pi(-1)^l}{\cos\pi\alpha} \sum_{m=0}^{l} \frac{(-l)_m (\frac{1}{2}-\alpha)_m}{m!\; m!} 2^m.$$

Here $(z)_m = z(z+1)\ldots(z+m-1)$; $(z)_0 = 1$.

When the numerical results are generated, the following approach is recommended for calculating $\phi_n(x)$

$$\int_a^b \frac{f(t)\mathrm{d}t}{(a-t)^\beta} = \int_b^{a-\varepsilon} \frac{f(t)\mathrm{d}t}{(a-t)^\beta} + \int_{a-\varepsilon}^{a} \frac{f(t)\mathrm{d}t}{(a-t)^\beta} \approx \int_b^{a-\varepsilon} \frac{f(t)\mathrm{d}t}{(a-t)^\beta} + f(a)\frac{\varepsilon^{1+\beta}}{1+\beta}.$$

This approach permits the avoidance of singularities, and gives accurate results when ε is small.

Now the interval $-1 \le x \le 1$ can be divided into N arbitrary finite sections and the coefficients C_n can be chosen in such a way that (3) will become an identity at all the nodal points x_0, x_1,..., x_N. This leads to the following system of $N+1$ linear algebraic equations for the determination of the unknown coefficients C_n

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{n-1} B_l x_i^{n-l-1} + \sum_{n=0}^{N} C_n \phi_n(x_i) = G(x_i), \quad i = 0,1,\ldots,N. \tag{7.1.4}$$

When $a=0$, (1) represents the integral equation of the first kind and therefore $\alpha = 0$. Eq. (4) will then simplify to

$$b\sum_{n=1}^{N} C_n \sum_{l=0}^{[(n+1)/2]} \sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)}\,x_i^{n-2l-1} + \sum_{n=0}^{N} C_n \phi_n(x_i) = G(x_i),$$
$$i = 0,1,\ldots,N.$$

Here $[r]$ denotes an integer not exceeding r.

When the solution of (1) equals zero at $x = \pm 1$, one can use the following form of approximate solution

$$f(x) = (1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}\sum_{n=0}^{N} C_n\, x^n. \tag{7.1.5}$$

Substitution of solution (5) into (1) yields

$$\frac{\pi b}{\cos\pi\alpha}\sum_{n=0}^{N} C_n\left[\left(\frac{1}{2}-2\alpha^2\right)\sum_{l=0}^{n-1} D_l x^{n-l-1} + x^n(2\alpha - x)\right] + \sum_{n=0}^{N} C_n \psi_n(x) = G(x). \tag{7.1.6}$$

Here the following integrals have been employed

$$\int_{-1}^{1} (1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha} x^k\, \mathrm{d}x = 2(-1)^k B(\tfrac{3}{2}+\alpha, \tfrac{3}{2}-\alpha)\, F(-k, \tfrac{3}{2}-\alpha; 3; 2),$$

$$\int_{-1}^{1} \frac{(1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}}{x-y}\, \mathrm{d}x = \frac{\pi}{\cos\pi\alpha}(2\alpha - y) - \pi\tan\pi\alpha\,(1+y)^{1/2-\alpha}(1-y)^{1/2+\alpha}.$$

Further,

$$\psi_n(x) = \int_{-1}^{1} k(x,t)(1+t)^{1/2-\alpha}(1-t)^{1/2+\alpha}\, t^n\, \mathrm{d}t,$$

$$D_l = (-1)^l \sum_{m=0}^{l} \frac{(-l)_m(\tfrac{3}{2}-\alpha)_m}{(3)_m m!}\, 2^m. \tag{7.1.7}$$

As previously, it can be stated that $a = 0$ leads to $\alpha = 0$, and the approximate solution will have the form

$$f(x) = \sqrt{1 - x^2} \sum_{n=0}^{N} C_n x^n. \tag{7.1.8}$$

In this case (6) will be simplified to the form

$$b \sum_{n=0}^{N} C_n \sum_{l=0}^{[(n+1)/2]} D_{2l} x^{n+1-2l} + \sum_{n=0}^{N} C_n \psi_n(x) = G(x). \tag{7.1.9}$$

All the D_k become zero when k is odd, and for even values of k, D_k becomes

$$D_{2l} = + \frac{\sqrt{\pi}}{2} \frac{\Gamma(l - 1/2)}{\Gamma(l + 1)}.$$

By substituting x_0, x_1, ..., x_N instead of x in (6) or (9), a system of $N + 1$ linear algebraic equations can be obtained for the determination of the coefficients C_n in the solution of singular integral equations of the first and second kind respectively.

Special consideration is necessary for the case when $k(x,t) = 0$, and the solution is unbounded at the points $x = \pm 1$, because C_0 becomes arbitrary as function $C_0(1 + x)^{-1/2-\alpha}(1 - x)^{-1/2+\alpha}$ is the solution of the homogeneous singular integral equation. Usually one has an additional equation for the determination of C_0, for example,

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = P, \tag{7.1.10}$$

which in the case of contact problems means that the integral of stresses under the die must be equal to the resulting force P.

Substitution of (2) into (10) yields after integration

$$\sum_{l=0}^{N} B_l C_l = P.$$

Hence,

$$C_0 = \frac{1}{B_0}\left[P - \sum_{l=1}^{N} B_l C_l \right]. \tag{7.1.11}$$

For the determination of the remaining N coefficients C_l, instead of (4), there will be a system of N linear algebraic equations, namely,

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{n-1} B_l x_i^{n-l-1} = G(x_i), \quad i = 1,2,\ldots,N. \tag{7.1.12}$$

And in the case $\alpha = 0$ the system of linear algebraic equations takes the form

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{[(n-1)/2]} \sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)} x_i^{n-1-2l} = G(x_i), \quad i = 1,2,\ldots,N. \tag{7.1.13}$$

Simplified approximate solution of singular integral equations. The method described in the previous section is to be used when a very accurate solution is needed in the whole interval $-1 \le x \le 1$ including edge points. However, as is shown in the following sections, the method can be simplified further so that the solution can be obtained readily even without employing a computer. Some accuracy may be sacrificed at points close to the edge points but the solution is sufficiently accurate at all other points.

Suppose the solution of (1) can be represented in the form,

$$f(x) = \sum_{n=0}^{N} \frac{C_n x^n}{\sqrt{1-x^2}}. \tag{7.1.14}$$

Substitution of solution (14) into (1) leads to the following system of $N+1$ linear algebraic equations for the determination of the coefficients C_n,

$$\frac{aC_0}{\sqrt{1-x_i^2}} + \sum_{n=1}^{N} C_n \left[\frac{a x_i^n}{\sqrt{1-x_i^2}} + b \sum_{l=0}^{[(n-1)/2]} \sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)} x_i^{n-1-2l} \right]$$
$$+ \sum_{n=0}^{N} C_n \chi_n(x_i) = G(x_i), \quad i = 0,1,\ldots,N. \tag{7.1.15}$$

Here,

$$\chi_n(x) = \int_{-1}^{1} \frac{k(x,t)t^n\,\mathrm{d}t}{\sqrt{1-t^2}},$$

and the ratio of the gamma functions can be given in the form

$$\sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)} = \pi\,\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots(l-\frac{1}{2})}{1\cdot 2\cdot 3\cdots l}, \tag{7.1.16}$$

and i represents the nodes in the division of the interval $-1 \le x \le 1$ into N sections.

For the case $k(x,t)=0$ one additional arbitrary constant is needed to satisfy the additional condition imposed by (10), in which case, the approximate solution is to be represented in the form

$$f(x) = \sum_{n=0}^{N} \frac{C_n x^n}{\sqrt{1-x^2}} + \frac{C}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}}. \tag{7.1.17}$$

The following formula may be obtained for the determination of C after substituting solution (17) into (10) and integrating.

$$C = \frac{\cos\pi\alpha}{\pi}\left[P - \sum_{l=0}^{[N/2]} C_{2l}\sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)}\right]. \tag{7.1.18}$$

Here again coefficients C_n are to be determined from the system of $N+1$ linear algebraic equations

$$\frac{aC_0}{\sqrt{1-x_i^2}} + \sum_{n=1}^{N} C_n\left[\frac{a x_i^n}{\sqrt{1-x_i^2}} + b\sum_{l=0}^{[(n-1)/2]} \sqrt{\pi}\frac{\Gamma(l+1/2)}{\Gamma(l+1)}\,x_i^{n-1-2l}\right] = G(x_i),$$

$$i = 0,1,\ldots,N. \tag{7.1.19}$$

It will be shown later that this approach gives reliable results not only for the case when the solution $f(x)$ is unbounded, but also when $f(x)$ is equal to zero at points $x=\pm 1$. The results are valid for the whole interval except at edge points close to $x=\pm 1$.

In the examples which follow it is shown that when the number of points of collocation is 11, numerical results are accurate enough in 95% of the interval. It is possible to look for a solution bounded at $x = \pm 1$ in the form

$$f(x) = \sqrt{1-x^2} \sum_{n=0}^{N} C_n x^n. \tag{7.1.20}$$

The system of $N+1$ linear algebraic equations in this case is

$$a \sum_{n=0}^{N} C_n x_i^n \sqrt{1-x_i^2} + b \sum_{n=0}^{N} C_n \frac{\sqrt{\pi}}{2} \sum_{l=0}^{[(n+1)/2]} \frac{\Gamma(l-1/2)}{\Gamma(l+1)} x_i^{n-2l+1}$$

$$+ \sum_{n=0}^{N} C_n \left[\chi_{n+2}(x_i) - \chi_n(x_i)\right] = G(x_i), \quad i = 0,1,...,N. \tag{7.1.21}$$

from which the coefficients C_n are evaluated.

Example 1. As the first example consider the case where $k(x,t)=0$ and $G(x)=\text{constant}$. Then,

$$f(x) + \frac{\cot \pi\alpha}{\pi} \int_{-1}^{1} \frac{f(t)\,\mathrm{d}t}{t-x} = E.$$

This equation corresponds to the plane contact problem of an inclined punch acting on an elastic half-plane with friction where $f(x)$ is the stress at the contact area, E is proportional to the angle of inclination and α is dependent on the coefficient of friction and elastic constants of the material of the half-plane.

The exact solution of this equation is given by Muskhelishvili (1953)

$$f(x) = \frac{E \sin\pi\alpha\,(x+2\alpha) + P(\cos\pi\alpha)/\pi}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}}.$$

The approximate solution was represented in (17). The coefficients C_n were determined by solving the systems of equations (19), and C was calculated using (18). The results for the case where E = 1, P = 1, are shown in Table 7.1.1. The results agree with the exact results within 3%.

Example 2. In this example, we consider the case where $k(x,t) = 0$ and $G(x)$ is a linear function of x, giving

$$f(x) + \frac{\cot \pi\alpha}{\pi} \int_{-1}^{1} \frac{f(t)\,dt}{t-x} = 2\alpha - x. \qquad (7.1.22)$$

The exact solution is bounded in $x = \pm 1$ and has the form

$$f(x) = \sin\pi\alpha\,(1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}.$$

This equation corresponds to the plane contact problem of paraboloidal punch acting on an elastic half-plane.

The approximate solution was sought using equations (17) and (20). In this case, P is fixed and equals $\pi(0.5 - 2\alpha^2)\tan \pi\alpha$. The results using both methods are shown in Table 7.1.2. The approximate solution using (17) agrees with the exact solution to 3 significant figures except at the point of collocation x_i = 0.95 where the agreement is within 2%. The other approximate solution using (20) agrees with the exact solution within 1% except at the points of collocation of $x_i = \pm 0.95$ where agreement is within 5%.

Table 7.1.1. Comparison of exact and approximate results in Example 1; x_i = points of collocation, C = coefficient (17), C_n = coefficients, f_a = approximate solution, f_e = exact solution.

α	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
	f_e	0.328	0.244	0.257	0.286	0.322	0.365	0.418	0.489	0.598	0.818	1.46
0.1	f_a	0.307	0.244	0.256	0.286	0.321	0.364	0.417	0.489	0.597	0.818	1.41
	C_n	0.247	0.259	–0.062	–0.004	0.064	–0.059	–0.436	0.132	0.820	–0.115	–0.539
	C	0.117										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.2	f_e	–0.438	0.0579	0.231	0.333	0.415	0.493	0.574	0.670	0.801	1.03	1.62
	f_a	–0.458	0.0573	0.229	0.331	0.414	0.491	0.572	0.668	0.796	1.03	1.52
	C_n	0.577	0.355	–0.197	–0.017	0.059	–0.176	–0.635	0.392	1.20	–0.398	–0.840
	C	–0.086										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.3	f_e	–0.923	0.081	0.354	0.491	0.589	0.673	0.754	0.843	0.955	1.14	1.54
	f_a	–0.937	0.078	0.351	0.488	0.586	0.670	0.751	0.839	0.949	1.13	1.45
	C_n	0.810	0.320	–0.345	–0.026	–0.019	–0.192	–0.461	0.421	0.872	–0.383	–0.686
	C	–0.14										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.4	f_e	–0.614	0.395	0.628	0.733	0.803	0.859	0.911	0.964	1.03	1.12	1.30
	f_a	–0.625	0.390	0.625	0.731	0.801	0.857	0.908	0.961	1.02	1.11	1.26

	C_n	0.952	0.187	–0.458	–0.020	–0.085	–0.115	–0.270	0.250	0.496	–0.233	0.477
	C	–0.095										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.49	f_e	0.771	0.929	0.961	0.975	0.983	0.990	0.995	1.00	0.101	1.02	1.03
	f_a	0.769	0.928	0.961	0.974	0.983	0.989	0.995	1.00	0.101	1.02	1.03
	C_n	1.00	0.0196	–0.500	–0.002	–0.109	–0.012	–0.199	0.026	0.354	–0.025	–0.394
	C	–0.01										

Example 3. As an illustration on the applicability of the simplified method to obtain an approximate solution for singular integro-differential equations, the following is considered.

$$f(x) + \frac{\cot \pi\alpha}{\pi} \int_{-1}^{1} \frac{f'(t)\,\mathrm{d}t}{t - x} = 10 \cos x.$$

Table 7.1.2. Comparison of exact and two approximate solutions in Example 2; x_i = points of collocation, f_e = exact solution, C_n, C = coefficients, f_{a1} = approximate solution (eq.(17)), f_{a2} = approximate solution (eq.(20))

α	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	0.139	0.231	0.284	0.308	0.315	0.309	0.291	0.260	0.215	1.49	0.0669
	f_{a1}	0.141	0.231	0.284	0.308	0.315	0.309	0.291	0.260	0.215	0.149	0.0668
0.1	C_{n1}	0.320	–0.064	–0.303	0.040	–0.0034	–0.0095	0.0067	–0.0027	–0.014	0.0087	0.0072
	f_{a2}	0.132	0.230	0.283	0.307	0.314	0.308	0.290	0.259	0.214	0.148	0.063
	C_{n2}	0.308	–0.062	0.0018	–0.016	0.061	–0.055	–0.291	0.113	0.556	–0.119	–0.336
	C	–0.011										
	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	0.382	0.547	0.620	0.638	0.625	0.588	0.531	0.455	0.356	0.227	0.088
	f_{a1}	0.389	0.547	0.621	0.638	0.625	0.588	0.531	0.455	0.356	0.227	0.089
0.2	C_{n1}	0.600	–0.24	–0.539	0.147	–0.019	0.041	0.016	–0.023	–0.040	0.046	0.012
	f_{a2}	0.360	0.544	0.617	0.636	0.622	0.586	0.529	0.453	0.354	0.226	0.083
	C_{n2}	0.586	–0.235	0.041	–0.067	0.101	–0.199	–0.316	0.405	0.587	–0.465	–0.254
	C	–0.012										
	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	0.758	0.938	0.981	0.956	0.895	0.809	0.702	0.575	0.427	0.251	0.0842
	f_{a1}	0.771	0.938	0.981	0.956	0.895	0.809	0.702	0.575	0.427	0.251	0.0861
0.3	C_{n1}	0.881	–0.491	–0.661	0.289	–0.049	0.096	0.0002	–0.076	–0.024	0.124	0.028
	f_{a2}	0.716	0.934	0.976	0.953	0.853	0.807	0.700	0.573	0.425	0.250	0.080
	C_{n2}	0.807	–0.485	0.138	–0.144	0.163	–0.532	–0.103	1.16	0.016	–1.32	0.462
	C	–0.008										

	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	1.29	1.37	1.32	1.22	1.10	0.951	0.792	0.621	0.437	0.237	0.0686
	f_{a1}	1.30	1.37	1.32	1.22	1.10	0.951	0.792	0.621	0.437	0.237	0.0705
0.4	C_{n1}	0.955	–0.764	–0.645	0.416	–0.086	0.176	–0.058	–0.189	0.078	0.262	–0.149
	f_{a2}	1.24	1.37	1.32	1.22	1.09	0.950	0.791	0.620	0.436	0.236	0.0663
	C_{n2}	0.950	–0.763	0.297	–0.195	2.20	–1.45	0.661	0.342	–2.02	–3.56	2.58
	C	–0.004										
	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	1.88	1.76	1.58	1.39	1.19	1.00	0.803	0.605	0.405	0.204	0.0518
	f_{a1}	1.88	1.76	1.58	1.39	1.19	1.00	0.803	0.605	0.405	0.204	0.0522
0.5	C_{n1}	1.00	–0.979	–0.520	0.479	–0.108	0.273	–0.179	0.374	0.314	0.450	–0.362
	f_{a2}	1.87	1.76	1.58	1.39	1.19	0.999	0.803	0.605	0.405	0.204	0.0515
	C_{n2}	0.999	–0.989	0.487	–0.123	0.047	–3.41	3.06	8.37	–7.75	–8.05	7.51
	C	0.0003										

The approximate solution of the equation is sought in the form

$$f(x) = \sqrt{1-x^2}\sum_{n=0}^{N} C_n x^n.$$

The coefficients C_n were determined from the system of linear algebraic equations,

$$\sum_{n=0}^{N} C_n x_i^n \sqrt{1-x_i^2} + \frac{\cot \pi\alpha}{\pi}\sum_{n=0}^{N} C_n \sum_{l=0}^{[n/2]} \sqrt{\pi}\left(\frac{n+1}{2} - l\right)\frac{\Gamma(l-1/2)}{\Gamma(l+1)} x_i^{n-2l} = 10\cos x_i.$$

The exact solution of the singular integral equation under consideration is not known at this time. As a partial check on the validity of the method two different sets of points of collocations $N = 10$ and $N = 20$ were used in the calculations in order to investigate the convergence of the procedure. The results are presented in Table 7.1.3. A comparison of f_1 and f_2 proves good convergence of the procedure proposed.

Table 7.1.3. Convergence of two approximate solutions of singular integro-differential equations in Example 3; x_i = points of collocations, f_1 = solution for $N = 20$, f_2 = solution for $N = 10$.

α	x	–0.95	–0.80	–0.60	–0.40	–0.2	0.0	0.20	0.40	0.60	0.80	0.95
0.1	f_1	–1.06	–2.18	–3.10	–3.71	–4.06	–4.18	–4.06	–3.71	–3.10	–2.18	–1.06
	f_2	–1.06	–2.18	–3.10	–3.71	–4.06	–4.18	–4.06	–3.71	–3.10	–2.18	–1.06
0.2	f_1	–4.34	–9.15	–13.26	–16.03	–17.67	–18.21	–17.67	–16.03	–13.26	–9.15	–4.34
	f_2	–4.35	–9.15	–13.26	–16.03	–17.67	–18.22	–17.67	–16.03	–13.26	–9.15	–4.35

0.3	f_1	14.50	32.17	48.48	60.05	67.10	69.47	67.10	60.05	48.48	32.17	14.50
	f_2	14.53	32.15	48.46	60.03	67.07	69.45	67.07	60.03	48.46	32.15	14.53
0.4	f_1	1.41	4.66	9.92	15.26	19.24	20.71	19.24	15.26	9.92	4.66	1.41
	f_2	1.42	4.66	9.92	15.26	19.24	20.71	19.24	15.26	9.92	4.66	1.42
0.49	f_1	11.75	6.94	8.49	9.48	10.09	10.30	10.09	9.48	8.49	6.94	11.75
	f_2	8.70	6.98	8.55	9.48	10.10	10.30	10.10	9.48	8.55	6.98	8.70

Tables 7.1.1 and 7.1.2 show that the simplified method gives good results with a small number (eleven) of points of collocations. It is noteworthy that results are accurate even in the case when the form of the approximate solution (17) contradicts the condition that the solution is to be bounded in $x = \pm 1$, as it was in (22). The approximate solution (17) contains a singularity of ∞ at the edges $x = \pm 1$ whereas (22) asks for a singularity of 0 at the edges. Values of C calculated for a wide range of α are very small and may be neglected (Table 7.1.2). The simplified method can be use easily by practicing engineers. The method also proved to be stable with respect to the choice of the points of collocations. When good accuracy is necessary at the points very close to $x = \pm 1$, the more exact general approach (4) must be applied.

7.2. One-dimensional integro-differential equations

A method for the numerical solution of singular integro-differential equations is proposed. The approximate solution is sought in the form of the sum of a power series with unknown coefficients multiplied by a special term which controls the appropriate solution behavior near and at the edges of the interval. The coefficients are to be determined from a system of linear algebraic equations. The method is applied to the solution of a contact problem of a disk inserted in an infinite elastic plane. Exact analytical solution is obtained for the particular case when the disk is of the same material as the plane. Comparison is made between the exact and the approximate solution as well as with the solutions previously available in literature. The stability and the accuracy of the present method is investigated under variation of the parameters involved. The application of the method to the case when the boundary conditions for the unknown function are nonzero is discussed along with an illustrative example. A FORTRAN subroutine for the numerical solution of singular integro-differential equations is also provided.

Applications of singular integro-differential equations in various branches of mechanics are well known. Among these are elastic contact problems, stresses in composite materials, airfoil problems, etc. The exact solution of these equations is usually not available and in such cases approximate methods have been commonly used. As early as in 1938 Multhopp suggested a method using Chebyshev polynomials and applied it to the solution of the Prandtl type equation. Later the convergence of Multhopp's method was established by Kalandiia (1957). Methods for the reduction of singular equations to systems of linear algebraic equations by use of different approximate integration rules of Gauss, Radau, Lobatto, etc., were developed by Krenk (1975), Theocaris and Ioakimidis (1979). Methods using the properties of Chebyshev and Jacobi polynomials to obtain the exact evaluation of singular integrals were suggested by Morar and Popov (1976), and Erdogan (1969). In these investigations, the least square method is employed to obtain a system of linear algebraic equations for the determination of the unknown coefficients.

The method proposed in this section uses a simpler power series formulation instead of orthogonal polynomials and subsequently the collocation method (instead of the least square method) for obtaining the system of equations. Such a procedure simplifies the calculations considerably without sacrificing accuracy.

Description of the method. We consider the singular integro-differential equation

$$A(x)f(x) + A_1(x)f'(x) + B(x)\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{t-x} + B_1(x)\int_{-1}^{1}\frac{f'(t)\mathrm{d}t}{t-x}$$

$$+\int_{-1}^{1}K(x,t)\,f(t)\,\mathrm{d}t + \int_{-1}^{1}K_1(x,t)f'(t)\,\mathrm{d}t = G(x) \quad (-1\le x\le 1) \tag{7.2.1}$$

$$f(-1) = \mathrm{f}(1) = 0.$$

Here A, A_1, B, B_1, G are known continuous and differentiable functions; K, K are known kernels, continuous with respect to both variables x and t, and may have week singularities with respect to t at the points $t=\pm 1$. For simplicity equation (1) may be written as follows:

$$Lf(t) = G(x) \quad (-1\le x,t\le 1). \tag{7.2.2}$$

Here L denotes the left-hand operator of equation (1).

It is proven correct to use orthogonal polynomials for the solution of singular integro-differential equations. Since any linear combination of orthogonal polynomials is a polynomial of general form, one may seek an approximate solution of (1) in the form

$$f(x) = S_\delta(x)\sum_{n=0}^{N}C_n x^n, \tag{7.2.3}$$

where $S_\delta(x) = (1+x)^{1/2-\delta}(1-x)^{1/2+\delta}$, and C_n are coefficients to be determined. In general, $-0.5<\delta<0.5$; in the particular case when $A_1(x)/B_1(x)$=const the value of δ is

$$\delta = \frac{1}{\pi}\tan^{-1}\left(\frac{A_1}{\pi B_1}\right). \tag{7.2.4}$$

The influence of the value of δ on the accuracy of the solution is investigated further. The solution in the form (3) provides zero boundary values of the function f. The case of $f(\pm 1)\neq 0$ will be considered later.

Substitution of (3) into (1) yields

$$\sum_{n=0}^{N} C_n Y_n(x) = G(x)$$

where

$$Y_n(x) = A(x)S_\delta(x)x^n + [A_1(x) - \pi B_1(x)\tan(\pi\delta)]\,\frac{n-2\delta x-(n-1)x^2}{(1+x)^{1/2+\delta}(1-x)^{1/2-\delta}}\,x^{n-1}$$

$$+ B(x)\,\phi_n(x) + B_1(x)\,\xi_n(x) + R_n(x), \tag{7.2.5}$$

where

$$\phi_n(x) = \int_{-1}^{1}\frac{t^n(1+t)^{1/2-\delta}(1-t)^{1/2+\delta}}{t-x}\,\mathrm{d}t = \sum_{k=0}^{n-1} D_k x^{n-1-k}$$

$$-\frac{\pi}{\cos\pi\delta}(x-2\delta)\,x^n - \pi x^n S_\delta(x)\tan\pi\delta$$

$$\phi_n'(x) = \int_{-1}^{1}\frac{[t^n(1+t)^{1/2-\delta}(1-t)^{1/2+\delta}]'}{t-x}\,\mathrm{d}t, \tag{7.2.6}$$

$$\xi_n(x) = \sum_{k=0}^{n-2}(n-1-k)\,D_k\,x^{n-2-k}, \tag{7.2.7}$$

$$D_k = \frac{\pi(1-4\delta^2)}{2\cos\pi\delta}\,{}_2F_1(-k, \tfrac{3}{2}+\delta, 3; 2), \tag{7.2.8}$$

$$R_n(x) = \int_{-1}^{1} K(x,t)\, S_\delta(t) t^n \,\mathrm{d}t + \int_{-1}^{1} K_1(x,t)[S_\delta(t) t^n]' \mathrm{d}t.$$

As usual, ${}_2F_1$ denotes the Gauss hypergeometric function; since $(-k)$ is an integer and negative, ${}_2F_1$ represents a polynomial in δ of the highest power equal to k. When the ratio $A(x)/B(x) = \text{const}$, and the value of δ is defined by (4), equation (5) simplifies since the second term vanishes. When δ equals zero, equation (8) yields

$$D_k = 0 \quad \text{for odd} \quad k = 2m+1$$

and

$$D_k = \frac{\sqrt{\pi}}{2} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+2)}, \quad \text{for even} \quad k = 2m. \tag{7.2.9}$$

The solution involves the choice of an appropriate set of points of collocations x_i, i=0,1,...,N, such that equation (5) becomes an identity at every point x_i; hence the following system of N+1 linear algebraic equations will be obtained for the determination of the coefficients C_n:

$$\sum_{n=0}^{N} C_n Y_n(x_i) = G(x_i) \quad (i=0,1,...N). \tag{7.2.10}$$

The approximate solution $f(x)$ can be obtained from equation (3) once values of the coefficients C_n are known.

Assessment of accuracy. The accuracy of the approximate solutions is easily evaluated when the exact solution is known. However, the exact solution is usually not available and therefore it is necessary to evaluate the accuracy of the solution indirectly. We consider the error function

$$q(x) = \sum_{n=0}^{N} C_n Y_n(x) - G. \tag{7.2.11}$$

One can determine the average square error as

$$Q_s = \left\{ \int_{-1}^{1} [q(x)]^2 \, dx \right\}^{1/2}, \tag{7.2.12}$$

the maximum absolute error as

$$Q_a = \max \left| q(x) \right| \quad (-1 \le x \le 1), \tag{7.2.13}$$

and the maximum relative error as

$$Q_r = \max \left\{ \frac{|q(x)|}{\sum_{n=0}^{N} \left| C_n Y_n(x) \right| + \left| G(x) \right|} \right\} \quad (-1 \le x \le 1). \tag{7.2.14}$$

It is obvious that $q(x) \equiv 0$ means that the solution is exact but it is not at all obvious that one approximate solution $f_{a1}(x)$ is more accurate than another solution $f_{a2}(x)$ if, for example, $Q_{a1} < Q_{a2}$, i.e. this inequality does not imply $\max \left| f_e(x) - f_{a1}(x) \right| < \max \left| f_e(x) - f_{a2}(x) \right|$ where f_e denotes exact solution. That is why it is necessary to introduce several error parameters. If all three error parameters Q_s, Q_a, Q_r are decreasing simultaneously, that means a more and more accurate solution is obtained in most cases. The utilization of these error parameters will be illustrated through examples in later sections.

Application of the method. Here, the method described is applied to a plane contact problem. Exact solution to the problem is obtained in a particular case. The dependence of the accuracy of the approximate solution on variation of δ and location of the points of collocations is discussed.

Consider a disk of radius ρ_2 made of a material with elastic modulus E_2 and Poisson's ratio v_2. The disk is inserted in a hole of radius ρ_1 in an infinite elastic plane (Fig. 7.2.1) with elastic modulus E_1 and Poisson's ratio v_1. The difference $\varepsilon = \rho_1 - \rho_1$ is assumed to be of the same order as the elastic displacements; therefore one may assume that $\rho_1 = \rho_2 = \rho$ and that the polar angle of the hole ζ_1 is equal to the one of the disk $\zeta_2 = \zeta_1 = \zeta$. A force P_0 and a twisting moment M_0 are applied at the centre of the disk. It is assumed that in the contact region $-\alpha \le \zeta \le \alpha$ the normal radial stresses $\sigma(\zeta)$ and tangential stresses $\tau(\zeta)$ both exist, and the following relationship between them is valid: $\tau(\zeta) = \lambda\sigma(\zeta)$, where λ is the friction coefficient and is taken as a constant. It is appropriate to orient the polar axis Or so that it passes through the centre of contact region. Let P_r and P_t be the radial and tangential components of the force P_0, respectively. The angle between P_0 and P_r is denoted by β. For prescribed M_0, P_0, ρ, ε and elastic properties, the problem is to find the stress

Fig. 7.2.1. Plane contact problem of a disk inserted in a plate

distribution σ in the contact region and the size of the contact region 2α.

This problem was solved in (Morar' and Popov, 1976) by the orthogonal polynomial method. Here it will be shown that the solution to the problem by the method proposed here leads to much simpler calculations without sacrifices of accuracy. According to their results, the following singular integro-differential equation is to be solved:

$$-\gamma_1\sigma(x) + \gamma_1\lambda\frac{1+a^2x^2}{2a}\sigma'(x) - 2\lambda\int_{-1}^{1}\frac{\sigma(t)\,\mathrm{d}t}{t-x} - \frac{1+a^2x^2}{a}\int_{-1}^{1}\frac{\sigma'(t)\,\mathrm{d}t}{t-x}$$

$$+2\lambda a^2 b_1 - \gamma_2\rho^{-1}\left(P_r\frac{1-a^2x^2}{1+a^2x^2} + P_t\frac{2ax}{1+a^2x^2}\right) - \gamma_2 P_1 = \varepsilon\gamma_4\rho^{-1}. \tag{7.2.15}$$

Here,

$$b_1 = \int_{-1}^{1}\frac{t\sigma(t)\mathrm{d}t}{1+a^2t^2}; \quad P_1 = a\int_{-1}^{1}\frac{\sigma(t)\mathrm{d}t}{1+a^2t^2}, \tag{7.2.16}$$

$$P_r = 2a\rho\int_{-1}^{1}\frac{1-a^2t^2-2\lambda at}{(1+a^2t^2)^2}\sigma(t)\,\mathrm{d}t, \tag{7.2.17}$$

$$P_t = 2a\rho \int_{-1}^{1} \frac{\lambda(1 - a^2t^2) + 2at}{(1 + a^2t^2)^2}\, \sigma(t)\, \mathrm{d}t, \tag{7.2.18}$$

$$\gamma_1 = 2\pi\gamma_5[(1 - \kappa_1)(1 + \nu_1)E_2 - (1 - \kappa_2)(1 + \nu_2)E_1]$$

$$\gamma_2 = 4\gamma_5[\kappa_1(1 + \nu_1)E_2 + (1 + \nu_2)E_1], \quad \gamma_3 = 2\gamma_5 E_2(1 + \kappa_1)(1 + \nu_1),$$

$$\gamma_4 = 4\pi\gamma_5 E_1 E_2, \quad \gamma_5 = [(1 + \kappa_1)(1 + \nu_1)E_2 + (1 + \kappa_2)(1 + \nu_2)E_1]^{-1},$$

$$x = \frac{1}{a}\tan\frac{\zeta}{2}, \quad a = \tan\frac{\alpha}{2}, \quad \kappa_j = 3 - 4\nu_j, \quad \text{for } j = 1,2.$$

Substitution of the solution (3) into (15) leads to the following system of linear algebraic equations, analogous to (10):

$$\sum_{n=0}^{N} C_n \Big[-\gamma_1 S_\delta(x)\, x_i^n\Big] + \left[\frac{\gamma_1\lambda}{2} + \pi\tan\pi\delta\right]\frac{1 + a^2x^2}{a}\,\frac{n - 2\delta x_i - (n+1)x_i^2}{(1 + x_i)^{1/2+\delta}(1 - x_i)^{1/2-\delta}}\, x_i^{n-1}$$

$$-\, 2\lambda\phi_n(x_i) - \frac{1 + a^2x^2}{a}\,\xi_n(x_i) + 2\lambda a^2\eta_{n+1} - 2\, a\gamma_2\Big[(\chi_n - a^2\chi_{n+2}$$

$$-\, 2\lambda a\chi_{n+1})\,\frac{1 - a^2x_i^2}{1 + a^2x_i^2} + (\lambda\chi_n - \lambda a^2\chi_{n+2} + 2a\chi_{n+1})\,\frac{2ax_i}{1 + a^2x_i^2}\Big] - \gamma_2 a\eta_n = \varepsilon\gamma_4\rho^{-1},$$

$$\text{for } i = 0,1, \ldots, N. \tag{7.2.19}$$

Here functions S_δ, ϕ_n, ξ_n were determined by (3), (6), and (7), and the parameters η_n and χ_n can be expressed in elementary functions as follows:

$$\eta_n = \int_{-1}^{1} \frac{S_\delta(t)t^n\, \mathrm{d}t}{1 + a^2t^2} = \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{a^{2k+2}} D_{n-2-2k} - \frac{\pi(-a^2)^{-[n/2]-1}}{\cos\pi\delta}\,[\cos H$$

$$+\, a(-a^2)^{-m}\sin H + m(1 + 2\delta) - 1], \tag{7.2.20}$$

$$\chi_n = \int_{-1}^{1} \frac{S_\delta(t)t^n\, \mathrm{d}t}{(1 + a^2t^2)^2} = \sum_{k=0}^{[n/2]-2} \frac{(-1)^k(1 + k)}{a^{2k+4}} D_{n-4-2k} + \left[\frac{n}{2}\right]\frac{\pi(-a^2)^{-[n/2]-1}}{\cos\pi\delta}\,[\cos H$$

$$+ a(-a^2)^{-m} \sin H + m(1+2\delta) - 1] + \frac{\pi(-a^2)^{-[n/2]}}{2a\cos\pi\delta}\Bigg[\Bigg(a^{-2m}$$

$$-\frac{1+2\delta}{1+a^2}\Bigg)\sin H + \frac{a}{(-a^2)^m}\frac{1+2\delta}{1+a^2}\cos H\Bigg], \qquad (7.2.21)$$

where

$$H = (1+2\delta)\tan^{-1} a = [(1+2\delta)\alpha]/2, \quad m = n - 2[n/2],$$

D_k is determined by (8), and $[n/2]$ denotes a maximum integer not exceeding $n/2$.

The integral in (20) was computed by using formula 3.228 from (Gradshtein and Ryzhik, 1965), and the following representation was used:

$$\frac{1}{1+a^2t^2} = \frac{1}{2}\left[\frac{1}{1-iat} + \frac{1}{1+iat}\right].$$

Expression (21) was derived from (20) by means of differentiation of both sides of (20) with respect to a. The system (19) may be simplified by assuming

$$\delta = -\frac{1}{\pi}\tan^{-1}\left(\frac{\gamma_1 \lambda}{2\pi}\right). \qquad (7.2.22)$$

In this case the second term in the system (19) will vanish. Certain simplifications are then possible if $\delta = 0$. Besides obvious simplification (19), the expressions (20) and (21) will also be simplified hence for odd $n = 2l+1$, we have $\eta_n = \chi_n = 0$; and for even $n = 2l$ hold

$$\eta_{2l} = \int_{-1}^{1} \frac{\sqrt{1-t^2}\, t^{2l}}{1+a^2t^2}\, dt = -\frac{\sqrt{\pi}}{2}\sum_{k=0}^{l-1} \frac{\Gamma(k+1/2)}{\Gamma(k+2)}(-a^2)^{k-l}$$

$$+ \frac{(-1)^l \pi}{a^{2l+2}}(\sqrt{1+a^2} - 1), \qquad (7.2.23)$$

$$\chi_{2l} = \int_{-1}^{1} \frac{\sqrt{1-t^2}\, t^{2l}}{(1+a^2t^2)^2}\, dt = -\frac{\sqrt{\pi}}{2}\sum_{k=0}^{l-2} (k-l+1)\frac{\Gamma(k+1/2)}{\Gamma(k+2)}(-a^2)^{k-l}$$

$$+\frac{(-1)^l\pi}{2a^{2l}}\left[\frac{1}{\sqrt{1+a^2}}-2l\,\frac{\sqrt{1+a^2}-1}{a^2}\right]. \tag{7.2.24}$$

After obtaining the values of coefficients C_n from the system (19), all other parameters can be easily calculated as follows:

$$P_0=\sqrt{P_r^2+P_t^2},\quad M_0=2\lambda\rho^2 a\sum_{n=0}^{N} C_n\,\eta_n,$$

$$P_r=2a\rho\sum_{n=0}^{N} C_n(\chi_n-a^2\chi_{n+2}-2\lambda a\chi_{n+1})$$

$$P_t=2a\rho\sum_{n=0}^{N} C_n(\lambda\chi_n-\lambda a^2\chi_{n+2}+2a\chi_{n+1}),\quad \beta=\tan^{-1}(P_t/P_r). \tag{7.2.25}$$

It is obvious that greater force corresponds to greater contact angle α, and it is obvious that there exists a maximum contact angle α_m called the critical angle which can not be exceeded under action of any force. Its value can be obtained by equating to zero the determinant of the system of equations (19).

Before presentation of numerical results, it is worthwhile to note that in the case when the disk and the plate are made of the same material ($E_1=E_2$, $\nu_1=\nu_2$), equation (15) can be solved exactly as $\gamma_1=0$. Having the exact solution for this particular case, one can estimate the accuracy of the approximate solution.

In the case of $\gamma_1=0$, equation (15) can be transformed as follows:

$$\int_{-1}^{1}\frac{w(t)\,\mathrm{d}t}{x-t}=W(x),\qquad (-1\le x\le 1). \tag{7.2.26}$$

Here

$$w(t)=\frac{1+a^2t^2}{a}\,\sigma'(t)+2\lambda\sigma(t),\qquad \sigma(1)=\sigma(-1)=0. \tag{7.2.27}$$

$$W(x)=d_1\frac{1-a^2x^2}{1+a^2x^2}+d_2\frac{2ax}{1+a^2x^2}-d_3,\quad d_1=\gamma_2\rho^{-1}P_r,\quad d_2=\gamma_2\rho^{-1}P_t, \tag{7.2.28}$$

$$d_3 = \gamma_3 P_1 + \varepsilon\gamma_4\rho^{-1} + a \int_{-1}^{1} \left(1 - \frac{2\lambda at}{1 + a^2t^2}\right)\sigma(t)\,\mathrm{d}t. \tag{7.2.29}$$

The solution of equation (26) is known (Muskhelishvili, 1953), and has the form

$$w(t) = -\frac{1}{\pi^2\sqrt{1-t^2}} \int_{-1}^{1} \frac{W(x)\sqrt{1-x^2}\,\mathrm{d}x}{x-t} + \frac{d_4}{\pi\sqrt{1-t^2}}, \tag{7.2.30}$$

and d_4 is an arbitrary constant. Substitution of (28) into (30) yields

$$w(t) = \frac{d_1 t}{\pi\sqrt{1-t^2}}\left[1 - \frac{2\sqrt{1+a^2}}{1+a^2t^2}\right] + \frac{2d_2}{\pi\sqrt{1-t^2}}\left[\frac{\sqrt{1+a^2}}{1+a^2t^2} - 1\right]$$

$$-\frac{d_3 t}{\pi\sqrt{1-t^2}} + \frac{d_4}{\pi\sqrt{1-t^2}}. \tag{7.2.31}$$

Now equation (27) is an ordinary differential equation with respect to σ, and its exact solution can be written in a standard form:

$$\sigma(x) = \exp[-2\lambda\tan^{-1}(ax)]\left\{\int_{-1}^{x} \frac{aw(t)\exp[2\lambda\tan^{-1}(at)]}{1+a^2t^2}\,\mathrm{d}t + d_5\right\}. \tag{7.2.32}$$

From the condition $\sigma(-1) = 0$, it follows that $d_5 = 0$. It is not so easy to define other constants d_k, k=1,2,3, and hence they may be expressed through some integrals of σ. The parameter d_4 is to be determined from the condition $\sigma(1) = 0$. This is why Panasiuk and Teplyi (1972) determine the constants d_k approximately using the approximation of the exponents involved in (32) as linear functions. An exact solution for the particular case $\lambda = 1$ can be found in (Stippes *et. al.*, 1962).

Here, the exact solution is given for a general case. We introduce the following parameters:

$$Z_1 = \int_{-1}^{1} \frac{\sigma(t)\,\mathrm{d}t}{1+a^2t^2}, \quad Z_2 = \int_{-1}^{1} \frac{\sigma(t)\,\mathrm{d}t}{(1+a^2t^2)^2},$$

$$Z_3 = \int_{-1}^{1} \frac{\sigma(t)\, t\, \mathrm{d}t}{(1 + a^2t^2)^2}, \quad Z_4 = \int_{-1}^{1} \left(1 - \frac{2\lambda at}{(1 + a^2t^2)^2}\right)\sigma(t)\, \mathrm{d}t. \tag{7.2.33}$$

The constants d_k may be expressed through Z_k by comparison of (28), (29), (17), and (18) with (33). The result is

$$d_1 = \gamma_2[(2Z_2 - Z_1)2a - 4\lambda a^2 Z_3],$$

$$d_2 = \gamma_2[4a^2 Z_3 + 2a\lambda(2Z_2 - Z_1)],$$

$$d_3 = \gamma_3 a Z_1 + \varepsilon\gamma_4\rho^{-1} + aZ_4. \tag{7.2.34}$$

Direct calculation of Z_k using (33) and (32) is cumbersome, but we can express Z_k directly through some integrals of the function w, multiplying both parts of (27) by an appropriate term and integrating, with the result:

$$\int_{-1}^{1} \frac{w(t)\, \mathrm{d}t}{1 + a^2t^2} = 2\lambda Z_1, \quad \int_{-1}^{1} \frac{w(t)\, t\, \mathrm{d}t}{1 + a^2t^2} = -\frac{Z_4}{a},$$

$$\int_{-1}^{1} \frac{w(t)\, t\, \mathrm{d}t}{(1 + a^2t^2)^2} = \frac{1}{a}(Z_1 - 2Z_2 + 2\lambda a Z_3),$$

$$\int_{-1}^{1} \frac{w(t)\, \mathrm{d}t}{(1 + a^2t^2)^2} = 2\lambda Z_2 + 2aZ_3. \tag{7.2.35}$$

Substituting (31) in (35), we obtain, after integration,

$$2\lambda Z_1 = \frac{d_2}{a}\left[\frac{2 + a^2}{1 + a^2} - \frac{2}{\sqrt{1 + a^2}}\right] + \frac{d_4}{\sqrt{1 + a^2}},$$

$$Z_4 = \frac{d_1}{a\sqrt{1 + a^2}}\left[1 - \frac{1}{\sqrt{1 + a^2}}\right] + \frac{d_3}{a}\left[1 - \frac{1}{\sqrt{1 + a^2}}\right],$$

$$\frac{1}{a}(Z_1 - 2Z_2 + 2\lambda a Z_3) = \frac{d_1}{2(1+a^2)^{3/2}}\left[1 - \frac{4+a}{2\sqrt{1+a^2}}\right] - \frac{d_3}{2(1+a^2)^{3/2}},$$

$$2aZ_3 + 2\lambda Z_2 = \frac{d_2}{a\,(1+a^2)^{3/2}}\left[\frac{2+2a^2+(3/4)a^4}{\sqrt{1+a^2}} - 2 - a^2\right] + \frac{d_4(2+a^2)}{2(1+a^2)^{3/2}}. \tag{7.2.36}$$

Now considering together the system of equations (34) and (36), we can eliminate the Z-parameters, and obtain the following system of linear algebraic equations for the determination of the constants d_k, $k=1,2,3,4$:

$$d_1\left[1 + \frac{a^2\gamma_2}{(1+a^2)^{3/2}}\left(1 - \frac{4+a^2}{2\sqrt{1+a^2}}\right)\right] - d_3\frac{a^2\gamma_2}{(1+a^2)^{3/2}} = 0,$$

$$d_2\left[1 - \frac{\gamma_2}{(1+a^2)^{3/2}}\left(\frac{2+a^2+(1/2)a^4}{\sqrt{1+a^2}} - 2\right)\right] - d_4\frac{a\gamma_2}{(1+a^2)^{3/2}} = 0,$$

$$\frac{d_1}{\sqrt{1+a^2}}\left[-1 + \frac{1}{\sqrt{1+a^2}}\right] - d_2\frac{\gamma_3}{2\lambda}\left(\frac{2+a^2}{1+a^2} - \frac{2}{\sqrt{1+a^2}}\right)$$

$$+\frac{d_3}{\sqrt{1+a^2}} - d_4\frac{\gamma_3 a}{2\lambda\sqrt{1+a^2}} = \varepsilon\gamma_4\rho^{-1},$$

$$d_1 I_1 + d_2 I_2 - d_3 I_3 + d_4 I_4 = 0. \tag{7.2.37}$$

Here the fourth equation was obtained from the condition $\sigma(1) = 0$, and

$$I_1 = \frac{1}{\pi}\int_{-1}^{1}\frac{t}{\sqrt{1-t^2}}\left[1 - \frac{2\sqrt{1+a^2}}{1+a^2t^2}\right]\frac{\exp[2\lambda\tan^{-1}(at)]}{1+a^2t^2}\,dt,$$

$$I_2 = \frac{2}{\pi a}\int_{-1}^{1}\left[\frac{\sqrt{1+a^2}}{1+a^2t^2} - 1\right]\frac{\exp[2\lambda\tan^{-1}(at)]}{\sqrt{1-t^2}\,(1+a^2t^2)}\,dt,$$

$$I_3 = \frac{1}{\pi}\int_{-1}^{1}\frac{t\exp[2\lambda\tan^{-1}(at)]}{\sqrt{1-t^2}\,(1+a^2t^2)}\,dt,$$

$$I_4 = \frac{1}{\pi} \int_{-1}^{1} \frac{\exp[2\lambda \tan^{-1}(at)]}{\sqrt{1-t^2}\,(1+a^2t^2)}\,\mathrm{d}t. \tag{7.2.38}$$

Now the constants d_k are determined by (37), and together with expressions ((31) and (32) give the exact solution to equation (26). For the case of $\lambda = 0$, system (37) is not valid since for this case the first of equations (35) does not determine the value of Z_1 any more; instead, we shall have

$$Z_1 = \int_{-1}^{1} \frac{\sigma(t)\,\mathrm{d}t}{1+a^2t^2} = -\int_{-1}^{1} \frac{w(t)\tan^{-1}(at)\,\mathrm{d}t}{1+a^2t^2}.$$

The exact solution for $\lambda = 0$ can be expressed in terms of elementary functions and has the form

$$\sigma(x) = \frac{1}{2\pi\sqrt{1+a^2}} \left\{ 2d_1 \frac{a\sqrt{1-x^2}}{1+a^2x^2} + \left[d_1\left(1 - \frac{1}{\sqrt{1+a^2}}\right) - d_3 \right] \ln \frac{\left|a\sqrt{1-x^2} - \sqrt{1+a^2}\right|}{a\sqrt{1-x^2} + \sqrt{1+a^2}} \right\}. \tag{7.2.39}$$

Parameters $d_2 = d_4 = 0$, and the remaining d_1 and d_3 are determined from the set of equations

$$d_1\left[1 + \frac{a^2\gamma_2}{(1+a^2)^{3/2}}\left(1 - \frac{4+a^2}{2\sqrt{1+a^2}}\right)\right] - d_3 \frac{a^2\gamma_2}{(1+a^2)^{3/2}} = 0,$$

$$d_1\left[\left(\frac{1}{1+a^2} - \frac{1}{\sqrt{1+a^2}}\right)\left(1 - \frac{\gamma_3}{2}\ln(1+a^2)\right) - \frac{a^2\gamma_3}{2(1+a^2)}\right] + \frac{d_3}{\sqrt{1+a^2}}\left(1 - \frac{\gamma_3}{2}\ln(1+a^2)\right) = \varepsilon\gamma_4\rho^{-1}. \tag{7.2.40}$$

Here the following integrals were employed:

$$\int_{-1}^{1} \frac{t \tan^{-1}(at)\, \mathrm{d}t}{(1+a^2t^2)\sqrt{1-t^2}} = \frac{\pi \ln(1+a^2)}{2a\sqrt{1+a^2}},$$

$$\int_{-1}^{1} \frac{t \tan^{-1}(at)\, \mathrm{d}t}{(1+a^2t^2)^2\sqrt{1-t^2}} = \frac{\pi[\ln(1+a^2)+a^2]}{4a(1+a^2)^{3/2}}.$$

In order to compare the numerical results obtained by our method with those reported Morar' and Popov (1976), the calculations were made using the following values of parameters involved: $\nu_1=\nu_2=0.3$, $(E_2/E_1)=0.5, 1.0, 2.0, \infty$. All calculations were carried out assuming, according to (4), $\delta=-\tan^{-1}(\gamma_1\lambda/2\pi)$. All the results are in excellent agreement with those of Morar' and Popov, except for the value of critical angle of contact. To verify which method gives more accurate results, similar values were computed from the exact solution for $E_2=E_1$ by equating to zero the determinant of the system (37) for $\lambda\neq 0$. The same procedure was performed with the system (40) for $\lambda=0$. The results are presented in Table 7.2.1, where α_e denotes the exact value of the critical angle, α_a is the result of our numerical solution, α_M and α_E are the results of Morar' *et. al.* and Erdogan (1969) respectively.

Table 7.2.1. Comparison of values of the critical angle of contact (in degrees) obtained by our exact and approximate method with those reported by other authors

λ	0.0	0.05	0.10	0.15	0.20	0.30	0.40
α_e	169.66	169.66	169.69	169.73	169.79	169.95	170.18
α_a	169.66	169.66	169.69	169.73	169.79	169.95	170.18
α_M	169.6	–	–	–	170.0	171.2	–
α_E	169.66	170.02	171.01	172.50	174.33	–	–

In order to investigate the influence of δ on the accuracy of approximate solution, a comparison was made of the exact solution σ_e for $E_1=E_2$ (which was calculated by solving system (37), and using (31) and (32)), with approximate solution σ_a, corresponding to different values of δ. Since equation (4) in this case implies that $\delta=0$, the approximate solution for this case is identical to the exact one. As should be expected, the more δ differs from the one given by (4), the greater is the error of the approximate solution. The error parameters introduced in (12)–(14) were proven to be very sensitive to accuracy of the solution. When an approximate solution differs from the exact ones by 30%, the error parameters increase by the order of 10^5 times. A similar picture emerges for the common case $E_1 \neq E_2$, i.e. the best accuracy is obtained with the values of δ prescribed by (4).

It is also of interest to find out how strongly the choice of the points of collocations can affect the accuracy of solution. The test computations were performed with three sets of points: $\{x\}_1 = -\cos[(i+1/2)\pi/(N+1)]$, $i=0,1, \ldots, N$; $\{x\}_2 = -0.99$, -0.9, -0.7, -0.5, -0.25, 0.0, 0.25, 0.5, 0.7, 0.9, 0.99; $\{x\}_3 = -0.95$, -0.8, -0.6, -0.4, -0.2, 0.0, 0.2, 0.4, 0.6, 0.8, 0.95. All three error parameters were computed for various values of δ. All the calculations were performed for $E_1 = 2E_2$, $\lambda = 0.4$, $\alpha = 150$ degrees. The first set gives slightly better accuracy than the second one. The third set proved to be the least accurate, and its inaccuracy sharply increases when the value of δ does not correspond to the one given by (4).

Non-zero boundary conditions. Here we consider the case of singular integro-differential equation (1), with the following boundary conditions:

$$f(-1) = h_1, \quad f(1) = h_2. \tag{7.2.41}$$

We can look for an approximate solution in the form

$$f(x) = S_\delta(x) \sum_{n=0}^{N} C_n x^n + x(h_2 - h_1)/2 + (h_2 + h_1)/2, \tag{7.2.42}$$

with

$$S_\delta(x) = (1+x)^{(1/2)-\delta}(1-x)^{(1/2)+\delta}, \quad -\frac{1}{2} < \delta < \frac{1}{2}.$$

The solution (42) satisfies the boundary conditions (41) exactly, and its substitution in (1) leads to the solution of the same system of equations (10), with the only one difference: instead of $G(x)$ we should use

$$G^*(x) = G(x) - L\{x(h_2 - h_1)/2 + (h_2 + h_1)/2\}. \tag{7.2.43}$$

We consider as an example the following equation

$$f(x) + 1000\,f'(x) + \int_{-1}^{1} \frac{f(t)\,dt}{t-x} + \int_{-1}^{1} \frac{f'(t)\,dt}{t-x} + \int_{-1}^{1} xtf(t)\,dt = G(x), \qquad (7.2.44)$$

with

$$G(x) = \frac{14}{3} + 2008.4x + 3003x^2 + x^3 + (2x + 4x^2 + x^3)\ln\frac{|1-x|}{|1+x|},$$
$$f(-1) = 0, \quad f(1) = 2.$$

It is easy to verify that the exact solution of (44) is

$$f(x) = x^2 + x^3. \qquad (7.2.45)$$

The approximate solution has the form

$$f(x) = S_\delta(x) \sum_{n=0}^{N} C_n x^n + x + 1. \qquad (7.2.46)$$

Substitution of (46) into (1) leads to the system of equations (10) with

$$G^*(x) = x^3 + 3003x^2 + 2006\tfrac{11}{15}x - 998\tfrac{1}{3} + (x^3 + 4x^2 + x - 2)\ln\frac{|1-x|}{|1+x|}.$$

As in previous example, all the calculations were made for the same three sets of the points of collocations for different values of δ. The qualitative picture is the same: the first set is slightly better than the second, and considerably better than the third one. The highest accuracy is obtained for the value of δ implied by (4), namely,

$$\delta = \frac{1}{\pi}\tan^{-1}(1000/\pi) = 0.499$$

This example proves that the method is effective also for the case of non-zero boundary conditions. The method is more simple than the method of orthogonal polynomials, with no less accuracy in results. The method can use arbitrary points of collocations (not necessarily the roots of orthogonal polynomials), hence, there is no need for roots evaluation. The method allows us to compute the unknown function at arbitrary points directly, without cumbersome interpolation.

The software description. The simplicity of the method allowed us to develop an efficient FORTRAN subroutine for numerical solution of singular integro-differential equations. The procedure makes it possible to find an optimal set of points of collocations and an optimal value of δ. This optimum is indicated by the smallest values of the error parameters Q and by the convergence of the coefficients C_n.

The subroutine solves singular integro-differential equation (1), with zero boundary conditions. The case of non-zero boundary conditions is treated according to (42)–(43). The output of the subroutine gives not only the values of unknown function, but also the values of coefficients C_n, the error parameters Q_s, Q_a, and Q_r, and the abscissae x_a and x_r where the maximum absolute error Q_a and the maximum relative error Q_r occur. The knowledge of coefficients C_n allows us to compute the unknown function at arbitrary point, and convergence of those coefficients is an indirect indication of a good accuracy of the solution. The values of x_a and x_r shows the way of changing the set of points of collocations or indicate the necessity of including an additional point. The subroutine listing follows

```
      SUBROUTINE  IDIFEQ(A,A1,B,B1,R,G,D,X,NX,F,C,QS,QA,QR,XR)
C
C     COMPUTER                     –CDC/SINGLE
C
C     PURPOSE                      –APPROXIMATE  SOLUTION  OF  SINGULAR
C                                   INTEGRO-DIFFERENTIAL  EQUATIONS
C                                   THE  SOLUTION  IS  SOUGHT  IN
C                                   THE  FORM  OF  THE  SUM
C                                   OF  TERMS  X**J*(1–X)**(0.5+D)*(1+X)**(0.5–D),
C                                   0≤J≤NX–1,      –0.5<D<0.5
C
C     USAGE            CALL  IDIFEQ(A,A1,B,B1,R,G,D,X,NX,F,C,QS,QA,QR,XR)
C
C     ARGUMENTS        A,A1,B,B1   –COEFFICIENTS  OF  THE  SINGULAR  EQUATION
C                                   (ARE  TO  BE  DETERMINED
C                                   AS  EXTERNAL  FUNCTIONS)

C                      R           –RESULT  OF  INTEGRATION  OF
C                                   REGULAR  PART  OF  THE
C                                   EQUATION  WITH  THE  WEIGHT
C                                   X**J*(1–X)**(0.5+D)*(1+X)**(0.5–D),
C                                   IS  TO  BE  DETERMINED  AS
C                                   EXTERNAL  FUNCTION  R(J,X)

C                      G           –RIGHT-HAND  PART  OF  THE  EQUATION;
C                                   IS  TO  BE  DETERMINED  AS
C                                   EXTERNAL  FUNCTION  G(X)

C                      D           –POWER  PARAMETER  IN  THE
C                                   SOLUTION  SOUGHT

C                      X           –INPUT  VECTOR  OF  DIMENSION  NX
C                                   (POINTS  OF  COLLOCATIONS
C                                   IN  THE  INTERVAL  –1<X<1

C                      NX          –DIMENSION  OF  VECTORS  X,  C,  F  (NX<41)
```

```
C                   F          –OUTPUT VECTOR OF DIMENSION NX
C                              (SOLUTION OF EQUATION
C                              AT THE POINTS X(I))

C                   C          –OUTPUT VECTOR OF DIMENSION NX
C                              (COEFFICIENTS IN THE
C                              APPROXIMATE SOLUTION)

C                   QS         –AVERAGE SQUARE ERROR (OUTPUT)

C                   QA         –MAXIMUM ABSOLUTE ERROR (OUTPUT)

C                   QR         –MAXIMUM RELATIVE ERROR (OUTPUT)

C                   XA         –ABSCISSA OF MAXIMUM ABSOLUTE ERROR

C                   XR         –ABSCISSA OF MAXIMUM RELATIVE ERROR
C
C     REMARK                   –THE FOLLOWING PROCEDURES FROM
C                              LIBRARY IMSLIB ARE USED:
C                              LEQT2F, IQHSCU, DCSQDU.
C                              LEQT2F – SUBROUTINE FOR SOLVING
C                                   A SYSTEM OF LINEAR
C                                   ALGEBRAIC EQUATIONS
C                              IQHSCU –SUBROUTINE FOR
C                                   ONE-DIMENSIONAL QUASI-CUBIC
C                                   HER–ITE INTERPOLATION
C                              DCSQDU – SUBROUTINE FOR CUBIC
C                                   SPLINE QUADRATURE

C     ALL THESE SUBROUTINES CAN BE REPLACED BY EQUIVALENT
C
C
        EXTERNAL A,A1,B,B1,R,G
        DIMENSION X(1),F(1),C(1),WK(2000),CC(161,3),XX(161),
      1 YY(161),ZZ(161),AA(41,41)
        EQUIVALENCE (WK(1),CC(1,1)),(WK(500),XX(1)),(WK(700),YY(1))
        SQ(X)=(1.+X)**(.5–D)*(1.–X)**(.5+D)
        SQ1(X)=(1.+X)**(.5+D)*(1.–X)**(.5–D)
C
C       CALCULATION OF THE COEFFICIENTS OF THE SYSTEM
C       OF LINEAR ALGEBRAIC EQUATIONS
C
        M=1 $ ID=6 $ PI=3.1415926
        DO 1 K=1,NX
        XK=X(K) $ IF(XK.EQ.–1.) XK=–.999 $ IF(XK.EQ.1.) XK=.999
        S=SQ(XK)
        DO 3 J=1,NX
        IF(J.EQ.1) GO TO 2
        S=S*XK
     2  IF(J.EQ.1)S2=–2.*D–XK
        IF(J.GT.1)S2=J–1–2.*D*XK–J*XK**2
        IF(J.GT.2)S2=S2*XK**(J–2)
        R3=A(XK)*S
```

```
      R5=(A1(XK)–B1(XK)*PI*TAN(PI*D)*S2/SQ1(XK)
      R2=B(XK)*FI(J–1,XK,D)
      R1=B1(XK)*SK(J–1,XK,D)
      R4=R(J–1,XK)
    3 AA(K,J)=R1+R2+R3+R4+R5
   20 FORMAT(1X,11G10.3)
    1 C(K)=G(XK)
C
C     SOLUTION OF THE SYSTEM OF LINEAR EQUATIONS
C
      CALL LEQT2F(AA,M,NX,41,C,ID,WK,IE)
      N=3*NX $ IF(N.LT.41)N=41
C
C     CALCULATION OF THE APPROXIMATE SOLUTION F(X)
C
      DO 4 K=1,NX
      XK=X(K) $ F(K)=C(1)
      DO 5 J=2,NX
    5 F(K)=F(K)+C(J)*XK**(J–1)
    4 F(K)=F(K)*SQ(XK)
C
C     CALCULATION OF THE ERRORS OF THE SOLUTION
C
      DO 6 K=1,N
      YY(K)=0.   $ XK=X(K)=(K–1)*2./(N–1)–1.
      IF(K.EQ.1)XK=XX(1)=–.999 $ IF(K.EQ.N)XK=XX(N)=.999
      ZZ(K)=ABS(G(XK)) $ S=SQ(XK)
      DO 7 J=1,NX
      IF(J.EQ.1)GO TO 8
      S=S*XK
    8 IF(J.EQ.1)S2=–2.*D–XK
      IF(J.GT.1)S2=J–1.–2.*D*XK–J*XK**2
      IF(J.GT.2)S2=S2*XK**(J–2)
      AA(1,1)=A(XK)*S+(A1(XK)–B1(XK)*PI*TAN(PI*D))*S2/SQ1(XK)+
       –20+B(XK)*FI(J–1,XK,D)+B1(XK)*SK(J–1,XK,D)+R(J–1,XK)
      YY(K)=YY(K)+C(J)*AA(1,1)
    7 ZZ(K)=ZZ(K)+ABS(C(J)*AA(1,1))
    6 YY(K)=YY(K)–G(XK)
      QA=YY(1) $ QR=YY(1)/ZZ(1) $ MY=MR=1.
      DO 9 K=2,N
      IF(ABS(YY(K)/ZZ(K)).LT.ABS(QR))GO TO 10
      QR=YY(K)/ZZ(K) $ MR=K
   10 IF(ABS(YY(K)).LT.ABS(QA))GO TO 9
      QA=YY(K) $ MY=K
    9 CONTINUE
      XA=XX(MY) $ XR=XX(MR)
      DO 11 K=1,N
   11 YY(K)=YY(K)**2
      CALL IQHSCU(XX,YY,N,CC,161,IER)
      CALL DCSQDU(XX,YY,N,CC,161,XX(1),XX(N),QS,IER)
      QS=SQRT(QS)
      RETURN
      END
C
```

```
C
C
      FUNCTION SK(N,X,D)
C
C     CALCULATION OF THE SINGULAR INTEGRAL
C     CORRESPONDING TO THE DERIVATIVE
C     OF THE TERM X**N*(1–X)**(.5+D)*(1+X)**(.5–D)
C
      PI=3.1415926
      S=0.  $ L=N–1 $ B=1.5+D $ C=3 $ Z=2 $ A1=2.–N
      SK=–PI/COS(PI*D)
      IF(N.EQ.0)GO TO 2
      SK=SK*2*(X–D)
      IF(N.EQ.1) GO TO 2
      IF(X.EQ.0.)GO TO 3
      DO 1 I=1,L
      K=I–1 $ A1=–K
    1 S=S+PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)*(N–1,–K)*
      *X**(N–2–K)
    4 SK=S–PI/COS(PI*D)*((N+1)*X–2.*N*D)*X**(N–1)
      GO TO 2
    3 S=PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)
      GO TO 4
    2 RETURN
      END

C
C
      FUNCTION F21(A,B,C,Z)
C
C     COMPUTATION OF THE GAUSS GYPERGEOMETRIC FUNCTION
C
      A1=A $ B1=B $ C1=C $ G=S=R=1.
    1 R=R*A1*B1*Z/C1/G
      S=S+R $ A1=A1+1.  $ B1=B1+1 $ C1=C1+1.  $ G=G+1.
      IF(R.EQ.0..AND.S.EQ.0.)GO TO 2
      IF(R.NE.0..AND.S.EQ.0.)GO TO 1
      IF(ABS(R/S).GT..000001)GO TO 1
    2 F21=S
      RETURN
      END
C
C
      FUNCTION FI(N,X,D)
C
C     CALCULATION OF THE SINGULAR INTEGRAL
C     CORRESPONDING TO THE TERM
C     X**N*(1–X)**(.5+D)*(1+X)**(.5–D)
C
      PI=3.1415926
      S=–PI*(TAN(PI*D)*(1–X)**(.5+D)*(1+X)**(.5–D)+
      +(X–2.*D)/COS(PI*D))
      FI=S
      IF(N.EQ.0)GO TO 2
```

```
      FI=FI*X+PI*(1.–4.*D*D)/2./COS(PI*D)
      B=1.5+D  $  C=3  $  Z=2
      IF(N.EQ.1)GO  TO  2
      S=S*X**N  $  A1=1.–N
      IF(X.EQ.0.)GO  TO  3
      DO  1  I=1,N
      K=I–1  $  A1=–K
    1 S=S+PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)*X**(N–1–K)
      GO  TO  4
    3 S=S+PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)
    4 FI=S
    2 RETURN
      END
```

7.3. Computer evaluation of two-dimensional singular integrals

An algorithm and a standard subroutine are developed for the evaluation of singular integrals over arbitrary two-dimensional domains. The integrand is a product of a Green's function with another function having a weak singularity at the boundary of the domain. Formulae are derived for an accurate estimation of the integral in the neighborhood of the singularities. The integral over the rest of the domain is evaluated by a library subroutine. The software developed is applied to a study of some contact problems of the theory of elasticity for non-classical domains.

There are numerous monographs on numerical integration. They discuss very thoroughly various methods of integration of non-singular functions, but deal very superficially with the problem of integration of singular ones. Though some theoretical results have been published, we are unaware of any standard subroutine available for the singular integrals evaluation. This lack of the standard software resulted in some cases in ignoring the singularities during computations which definitely undermined the accuracy of the numerical results. The development of such a procedure is the purpose of this section.

A method for the numerical integration of products of Green's functions with *non-singular* functions was reported in (Berger and Bernard, 1983). In many practical cases the second function is *singular* at the domain border. Their method is not applicable here. Another approach is required, and is discussed further. The approach is based on the formulae derived for the accurate estimation of the integral in the small neighborhood of the singularities. The integral over the remaining part of the domain is evaluated by a library subroutine. The software developed is checked against the case of an elliptical domain for which the exact solution is known, and an excellent accuracy is confirmed. In the last part, it is shown how the subroutines developed can be used for an approximate solution of some elastic contact problems for non-classical domains. The standard subroutines are presented in Appendix.

Theory. Consider a two-dimensional domain S. Let its boundary be given in polar coordinates as

$$\rho = a(\phi). \tag{7.3.1}$$

The following integral is encountered in various applications

$$I(N) = \iint_S \frac{\sigma(M)}{R(M,N)}\, dS, \tag{7.3.2}$$

where $R(M,N)$ is the distance between M and N which gives a singularity at N when $N \in S$; the function σ is also singular, and can be presented in polar coordinates in the form

$$\sigma(M) \equiv \sigma(\rho, \phi) = \frac{a(\phi)\, f(\rho, \phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}, \tag{7.3.3}$$

where the function f has no singularities in S. So, we have an integrand in (2) which has a singularity at $N \in S$ and a square root singularity along the border $\rho = a(\phi)$. Split the domain S into three subdomains (Fig. 7.3.1), namely, S_1 indicates a circular disk of radius ε having its centre at N; S_2 stands for a narrow closed strip $a(\phi)-\varepsilon < \rho < a(\phi)$; and S_3 denote the remaining part of S. It is assumed that $\varepsilon \ll a(\phi)$. The integral (2) over the subdomain S_3 is non-singular and can be evaluated by any standard subroutine. The problem lies in the evaluation of (2) over the subdomains S_1 and S_2. Here, we are to show that some formulae can be derived for the estimation of (2) over the subdomains S_1 and S_2 which are reasonably accurate for an almost arbitrary non-singular f. Two separate cases are to be considered, namely, *i*) when N is inside S (subdomains S_1 and S_2 do not intersect); *ii*) when N is at the border (singularities overlap) which is the most difficult. The case when $N \notin S$ does not require any special consideration.

Consider the first case. Let the polar coordinates of the points M and N be (ρ_0, ϕ_0) and (ρ, ϕ) respectively. The integrals to be evaluated are

$$I_1(\rho, \phi) = \iint_{S_1} \frac{a(\phi_0)\, f(\rho_0, \phi_0)}{[\, a^2(\phi_0) - \rho_0^2\,]^{1/2} [\, \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)\,]^{1/2}}\, dS, \tag{7.3.4}$$

Fig. 7.3.1. Subdomains of integration

$$I_2(\rho, \phi) = \iint_{S_2} \frac{a(\phi_0)\, f(\rho_0,\phi_0)}{[\, a^2(\phi_0) - \rho_0^2\,]^{1/2} [\, \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)\,]^{1/2}}\, \mathrm{d}S. \qquad (7.3.5)$$

Introduction of a local system of polar coordinates centered at N eliminates the singularity and allows a very simple estimation of I_1

$$I_1(\rho, \phi) \equiv 2\pi\varepsilon \frac{a(\phi)\, f(\rho,\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}. \qquad (7.3.6)$$

For the evaluation of I_2, rewrite (5) as

$$I_2(\rho, \phi) = \int_0^{2\pi} \mathrm{d}\phi_0 \int_{a(\phi_0)-\varepsilon}^{a(\phi_0)} \frac{a(\phi_0)\, f(\rho_0,\phi_0)\, \rho_0 \mathrm{d}\rho_0}{[\, a^2(\phi_0) - \rho_0^2\,]^{1/2} [\, \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)\,]^{1/2}}. \qquad (7.3.7)$$

Since ε is small and function f has no singularities at the border, an approximate integration with respect to ρ_0 can be performed, with the result

$$I_2(\rho, \phi) \approx (2\varepsilon)^{1/2} \int_0^{2\pi} \frac{a(\phi_0)^{3/2} f(a(\phi_0), \phi_0)\, d\phi_0}{[\,\rho^2 + a^2(\phi_0) - 2\rho a(\phi_0)\cos(\phi - \phi_0)]^{1/2}}. \tag{7.3.8}$$

Thus, the singular two-dimensional integral (5) has been reduced to a single integral with a regular integrand and can be evaluated by any standard subroutine. The formulae (6) and (8) not only give the necessary estimations but also indicate the order of the error if the singularities are ignored: it is ε in the neighborhood of N and it is of the order of $\sqrt{\varepsilon}$ at the boundary.

Consider now the case when N is at the border. Introduce the notation

$$\frac{d}{d\phi} a(\phi + 0) = a'_+ , \quad \frac{d}{d\phi} a(\phi - 0) = a'_-. \tag{7.3.9}$$

The most general is the case of an angular point at the border (Fig. 7.3.2).

Fig. 7.3.2. Geometry related to the derivation of (19) and (20)

The case of a smooth curve can be obtained from the general one by putting $a'_+ = a'_-$. Define the angles γ_+ and γ_- between the normal to the boundary and the polar radius as

$$\gamma_+ = -\frac{a'_+}{[\,a^2(\phi) + a'^2_+\,]^{1/2}}, \quad \gamma_- = \frac{a'_-}{[\,a^2(\phi) + a'^2_-\,]^{1/2}}. \tag{7.3.10}$$

The signs in (10) are chosen in such a way that the formula to be derived could be applied automatically to an arbitrary case, including the case of a smooth curve. It seems logical to represent the subarea S_1 as two rhombi, with the side equal ε and the angles at the apex N equal $\pi/2-\gamma_+$ and $\pi/2-\gamma_-$ respectively. Consider the estimation of the integral of the type (4) over a rhombus with the side ε and the acute angle $\pi/2-\gamma$. Since $\varepsilon \ll a(\phi)$, the following approximation is valid

$$a^2(\phi) - \rho^2 = [a(\phi) + \rho][a(\phi) - \rho] \approx 2a(\phi)\,[a(\phi) - \rho]. \tag{7.3.11}$$

Introducing a local system of polar coordinates (r,ψ) centered at N, the following relationship can be obtained

$$a(\phi) - \rho \approx r\sin\psi \,/\, \cos\gamma. \tag{7.3.12}$$

In the new system of coordinates, the integral in question can be rewritten in the form

$$\begin{aligned} I &\approx [\tfrac{1}{2}a(\phi)\cos\gamma]^{1/2} f(a(\phi),\phi) \int_0^{\pi/2-\gamma} d\psi \int_0^{b(\psi)} \frac{dr}{(r\sin\psi)^{1/2}} \\ &= [2a(\phi)\cos\gamma]^{1/2} f(a(\phi),\phi) \int_0^{\pi/2-\gamma} \left[\frac{b(\psi)}{\sin\psi}\right]^{1/2} d\psi, \end{aligned} \tag{7.3.13}$$

where $b(\psi)$ is the equation of rhombus in the local system of polar coordinates which has the form

$$b(\psi) = \begin{cases} \dfrac{\varepsilon\cos\gamma}{\cos(\psi+\gamma)} & \text{for} \quad 0 < \psi < \dfrac{\pi}{4} - \dfrac{\gamma}{2}; \\ \dfrac{\varepsilon\cos\gamma}{\sin\psi} & \text{for} \quad \dfrac{\pi}{4} - \dfrac{\gamma}{2} < \psi < \dfrac{\pi}{2} - \gamma. \end{cases} \tag{7.3.14}$$

Due to (14), the integral (13) can be rewritten as

$$I \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi)\cos\gamma \left[\int_0^{\pi/4-\gamma/2} \frac{d\psi}{[\sin\psi\cos(\psi+\gamma)]^{1/2}} + \int_{\pi/4-\gamma/2}^{\pi/2-\gamma} \frac{d\psi}{\sin\psi} \right]. \tag{7.3.15}$$

The substitution $\psi = (x-\gamma)/2$ reduces the first integral to

$$\int_{\gamma}^{\pi/2} \frac{dx}{[2(\sin x - \sin\gamma)]^{1/2}} = K\left(\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)\right), \qquad (7.3.16)$$

where K stands for the complete elliptic integral of the first kind. Formula (3.673) from (Gradshtein and Ryzhik, 1965) was used for the evaluation of (16). The second integral in (15) is elementary. Finally, evaluation of (15) yields

$$I \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi) \cos\gamma\, C(\gamma), \qquad (7.3.17)$$

where C is the coefficient which depends on the angle γ only

$$C(\gamma) = K\left(\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)\right) + \ln\left[\frac{1 + \cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}\right]. \qquad (7.3.18)$$

Since the subdomain S_1 consists of two rhombi, the estimation of (4) will take the form, according to (17) and (18),

$$I_1(\phi) \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi) \cos\gamma\, [\, C(\gamma_+) + C(\gamma_-)\,]. \qquad (7.3.19)$$

In the case of a smooth boundary $\gamma_+ = -\gamma_- = \gamma$, and formula (19) simplifies as follows

$$I_1(\phi) \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi) \cos\gamma\left[K\left(\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)\right) + K\left(\sin\left(\frac{\pi}{4} + \frac{\gamma}{2}\right)\right) + \ln\left(\frac{\cos(\gamma/2) + 1/\sqrt{2}}{\cos(\gamma/2) - 1/\sqrt{2}}\right)\right]. \qquad (7.3.20)$$

Formulae (6), (8), (19) and (20) give the necessary estimations, and we can proceed with the development of a standard subroutine for the singular integral evaluation.

The software description. The main subroutine is a real function SING(F,A,R,FI,E,ER,IE). The parameters are as follows. **F** is the nonsingular part of the real function to be integrated (should be provided by the user, expressed in the polar coordinates and should be specified in the calling program as EXTERNAL). **A** is a real function giving the boundary of the domain of integration (the same remark as previous). **R** and **FI** are the polar radius and the polar angle respectively of the field point (input), can be specified inside, outside or at the boundary of the domain of integration. **E** and **ER** are respectively the values of the desired (input) and the achieved (output) relative error. **IE** is the code of the error (output), according to the specifications of the standard library IMSL. The subroutines DCADRE and DBLIN from this library have been used for the evaluation of the regular part of the integral. The accuracy of the software developed was checked against the case when the domain of integration is an ellipse with the semiaxes ratio $b:a = 2:1$, and F = 1. The exact value of the integral in this case is known, and is $2\pi abK(k)$, where K stands for the complete elliptic integral of the first kind, and k is the ellipse eccentricity $k = (1 - b^2/a^2)^{1/2}$. All the computations were made with the relative error E = 0.00001. It was necessary to investigate the influence of the value of ε on the relative error of the integral computed. Of course, it is quite clear that the smaller the value of ε the more accurate is the result, but the problem is that we can not take ε arbitrary small for two reasons: increase of the computing time, and the IMSL subroutine DBLIN would not tolerate a very small ε and would, now and then, give us ’terminal error’. Here are some results. For $\varepsilon = 0.1$ the maximum relative error was about 1% for regular points, and it was about 7% for the points close to the boundary but not at the boundary. This phenomena is due to the rapid change in the denominator of (3) which reduces the accuracy of the estimation (5). The conclusion, one should make, is that in the case of evaluating the integral (2) very close to the boundary, it is advisable to evaluate the integral at the boundary and to make the necessary interpolation. Ignoring the singularity due to $1/R$ in (2) would lead to an error from 5% for the field points close to the center to 25% for the points close to the boundary. The integral over a narrow strip $a(\phi)-\varepsilon<\rho<a(\phi)$ is responsible for 25% to 35% of the value of the integral (2). Of course, the value 0.1 is too large for ε, and we took it in order to demonstrate that our procedure is sufficiently accurate even in this case, and also to demonstrate how large the error can be if the singularities are neglected. In the case $\varepsilon = 0.03$ the maximum error everywhere does not exceed 0.3%. The singularities are still responsible for up to 12% and 22% of the integral (2) respectively. Again, the portion due to the singularities increases with the movement of the field point closer to the boundary. In the case $\varepsilon = 0.001$ the result is accurate up to the fifth digit, and this value of ε was used in the subroutine SING. Even for such a small ε the singularities are still responsible for up to 1.2% and 12% respectively which means that the singularities should not be ignored. The listing of SING along with the accompanying subroutines is given in Appendix.

Application to contact problems. It is well known that the contact problem in the theory of elasticity is reduced to the solution of the integral equation

$$w(N) = H \int\int_S \frac{\sigma(M)}{R(M,N)}\, dS \qquad (7.3.21)$$

where w denotes the normal displacements under the punch (a known function), σ stands for the normal stress exerted by the punch (an unknown function), and H is a constant (see 5.1.9) which in the case of an isotropic elastic half-space takes on the value $H = (1-\nu^2)/\pi E$, ν and E being respectively the Poisson coefficient and the elasticity modulus.

The software developed allows us to obtain reasonably accurate solutions to various non-classical contact problems. Let us consider a flat-ended punch with an arbitrary planform S under the action of a centrally applied normal force P. Let the normal stress distribution under the punch be

$$\sigma = \frac{c\, a(\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}, \qquad (7.3.22)$$

where, as before $a(\phi)$ is the equation of the contact region in polar coordinates, and c is a constant which can be easily defined from the condition that the integral of σ over S should give the total force P

$$\int_0^{2\pi} d\phi \int_0^{a(\phi)} \frac{c\, a(\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}} \rho\, d\rho = c \int_0^{2\pi} a^2(\phi)\, d\phi = 2Ac = P, \qquad (7.3.23)$$

where A is the area of S. One gets immediately from (23) that

$$\sigma = \frac{P\, a(\phi)}{2A\, [\, a^2(\phi) - \rho^2\,]^{1/2}} \qquad (7.3.24)$$

For the case of a flat punch $w = const$. Now substituting (24) in (21), we can verify how close to a constant will be the displacements, produced by the stress distribution (24). Denote by w_0 the normal displacements at the center of the punch (ρ=0). One can get from (21)

$$w_0 = \frac{\pi H P A_1}{4A}, \qquad (7.3.25)$$

where

$$A_1 = \int_0^{2\pi} a(\phi)\, d\phi. \qquad (7.3.26)$$

Computations were made for three different punch planforms: a square, an oval and a 'shamrock'. The expression of $a(\phi)$ for the square with the edge $2l$ was taken in the form

$$a(\phi) = \begin{cases} \dfrac{l}{\cos\phi} & \text{for } -\pi/4 < \phi < \pi/4 \\ \dfrac{l}{\cos(\phi-\pi/2)} & \text{for } \pi/4 < \phi < 3\pi/4 \\ \dfrac{l}{\cos(\phi-\pi)} & \text{for } 3\pi/4 < \phi < 5\pi/4 \\ \dfrac{l}{\cos(\phi-3\pi/2)} & \text{for } 5\pi/4 < \phi < 7\pi/4 \end{cases} \tag{7.3.27}$$

for the oval

$$a(\phi) = [\, a_1^2\cos^2\phi + a_2^2\sin^2\phi \,]^{1/2}\,, \tag{7.3.28}$$

and for the 'shamrock'

$$a(\phi) = a_1 + a_2\cos 3x\,, \quad a_1 > a_2 > 0. \tag{7.3.29}$$

The case of a square punch has been considered before by several authors, the oval or the 'shamrock' punch seem never to have been considered before.

The results of computation of the dimensionless normal displacement $w^* = w/w_0$ against the dimensionless $x^* = x/l$ and $y^* = y/l$ are presented in Table 7.3.1.

Table 7.3.1. The normal displacements under a square punch.

y^* \ x^*	0.000	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	1.000
1.00	1.024	1.020	1.015	1.007	0.997	0.983	0.965	0.940	0.903	0.874	0.794
0.95	1.021	1.017	1.012	1.004	0.994	0.980	0.962	0.938	0.904	0.882	0.874
0.90	1.018	1.014	1.009	1.001	0.991	0.978	0.961	0.939	0.911	0.904	0.903
0.80	1.013	1.009	1.004	0.997	0.988	0.976	0.961	0.946	0.939	0.938	0.940
0.70	1.009	1.005	1.001	0.994	0.986	0.976	0.967	0.961	0.961	0.962	0.965
0.60	1.006	1.002	0.998	0.993	0.987	0.980	0.976	0.976	0.978	0.980	0.983
0.50	1.003	1.001	0.997	0.993	0.989	0.987	0.986	0.988	0.991	0.994	0.997
0.40	1.002	0.999	0.997	0.994	0.993	0.993	0.994	0.997	1.001	1.004	1.007
0.30	1.001	0.999	0.998	0.997	0.997	0.998	1.001	1.004	1.009	1.012	1.015
0.20	1.000	0.999	0.999	0.999	1.001	1.002	1.005	1.009	1.014	1.017	1.020
0.10	1.000	1.000	1.000	1.001	1.003	1.005	1.008	1.012	1.017	1.020	1.023
0.00	1.000	1.000	1.001	1.002	1.003	1.006	1.009	1.013	1.018	1.021	1.024

As one can see, at the major part of the square the displacements are very close to unity, the maximum error of 20% being achieved only at the apex, and the error decreases very rapidly with the distance from the apex. Taking into consideration that the sign of the error fluctuates, we may assume that the relationship between the total force P and the punch settlement δ, computed on the basis of (25), should be reasonably accurate. Introduce the following relationship

$$\delta = \frac{HP}{g\sqrt{A}}, \tag{7.3.30}$$

where g is a dimensionless coefficient

$$g = \frac{4\sqrt{A}}{\pi A_1}. \tag{7.3.31}$$

According to (26) and (27), one can easily compute the coefficient g for the square as

$$g = \frac{1}{\pi \ln(1+\sqrt{2})} = 0.3611,$$

which is very close to the value 0.3607 given by Maxwell for the capacitance of the square. Using the electrostatic analogy, one can easily deduce that our coefficient g is related to the capacity C of a flat lamina by

$$g = \frac{C}{\sqrt{A}}. \tag{7.3.32}$$

Of course, closeness to the result by Maxwell does not mean that our result is so accurate. The value of g which seems to be accurate was obtained in (Noble, 1960), and is 0.367, so that our result is in error by 1.6% which is not bad. The accuracy can be improved further by using the variational principle. According to Noble (1960), the following inequality is valid

$$g_e \geq g\left\{2 - \frac{1}{2A}\int_0^{2\pi} d\phi \int_0^{a(\phi)} \frac{w^*(\rho,\phi)\,\rho d\rho}{[\,a^2(\phi)-\rho^2\,]^{1/2}}\right\}, \tag{7.3.33}$$

where g_e stands for the exact value of the coefficient, g is the approximate value defined by (31). Since w^* has already been computed in Table 7.3.1, the additional computation due to (33) can be performed very easily, the singularity can be removed by the change of variables $\rho = a(\phi)\sin\psi$. The final result, according to (33), is $g_e \geq 0.365$ which is only 0.55% away from the correct value.

For the oval, defined by (28), one can compute

$$A = \frac{\pi}{2}(a_1^2 + a_2^2), \quad A_1 = 4a_1 E(k), \tag{7.3.34}$$

where it is assumed that $a_1 > a_2$; E stands for the complete elliptic integral of the second kind, and $k = (1 - a_2^2/a_1^2)^{1/2}$. The value of g can be estimated as

$$g = \frac{(2-k^2)^{1/2}}{\sqrt{2\pi}E(k)}. \tag{7.3.35}$$

For the numerical computations, we assumed $a_1 = 4$, $a_2 = 1$. in order to have an oval with a 'waist' which would be quite different from the elliptic shape. The results are presented in Table 7.3.2 as the dimensionless normal displacements $w^* = w/w_0$ against the polar angle ϕ and dimensionless radius $\rho^* = \rho/a(\phi)$.

Table 7.3.2. The normal displacements under the oval punch.

ϕ \ ρ^*	0	1/6	1/4	1/3	1/2	2/3	3/4	5/6	11/12	1.0
$\pi/2$	1	1.000	1.000	1.001	1.002	1.006	1.009	1.014	1.022	1.042
$11\pi/24$	1	1.000	1.000	1.001	1.002	1.005	1.008	1.010	1.010	1.004
$5\pi/12$	1	1.000	1.000	1.000	1.001	0.998	0.993	0.985	0.976	0.967
$3\pi/8$	1	1.000	1.000	0.999	0.994	0.981	0.974	0.966	0.958	0.951
$\pi/3$	1	1.000	0.999	0.996	0.987	0.973	0.967	0.961	0.956	0.951
$7\pi/24$	1	0.999	0.997	0.994	0.985	0.975	0.972	0.969	0.966	0.964
$\pi/4$	1	0.999	0.997	0.994	0.988	0.984	0.983	0.983	0.984	0.984
$5\pi/24$	1	0.999	0.998	0.996	0.995	0.997	0.999	1.001	1.004	1.007
$\pi/6$	1	1.000	0.999	0.999	1.003	1.010	1.015	1.019	1.024	1.029
$\pi/8$	1	1.000	1.002	1.004	1.012	1.023	1.029	1.036	1.042	1.049
$\pi/12$	1	1.001	1.004	1.007	1.019	1.033	1.040	1.048	1.056	1.064
$\pi/24$	1	1.002	1.005	1.010	1.023	1.039	1.048	1.056	1.065	1.074
0	1	1.002	1.005	1.011	1.025	1.041	1.050	1.059	1.068	1.077

As we can see, the positive error does not exceed 8% while the negative error is below 6%, the error being mainly restricted to a narrow strip close to the boundary. Computations, according to (33), give the lower bound $g_e \geq 0.3758$. Formula (35) gives $g = 0.3835$ which is about 2% above the lower bound. Since the negative error in Table 7.3.2 is below 6%, one can estimate the upper bound as 0.3946 which guarantees that formula (35) is accurate, at least, within 3%. We strongly believe that the real accuracy is much better than 3% but even 3% is not bad for such a simple formula. It is also clear that the value of error will decrease with a_1 approaching a_2 since formula (30) in the case of an ellipse is exact.

The following values can be computed for the ’shamrock’-shaped punch (29): $A=\pi(a_1^2+a_2^2/2)$, $A_1=2\pi a_1$, and the value of g will be defined as

$$g=\frac{2\,[\,\pi(a_1^2+a_2^2/2)\,]^{1/2}}{\pi^2 a_1} \tag{7.3.36}$$

The results of computations for $a_1=1.5$, $a_2=1$ are presented in Table 7.3.3. Here, the maximum positive error and the maximum negative error are about 5%. The value of g due to (36) is 0.3971 while the lower bound for g, according to (33), is 0.4006. Since the error fluctuates almost symmetrically around zero, there is a reason to believe that the exact value of g will be very close to the lower bound.

Table 7.3.3. The numerical results for the ’shamrock’-shaped punch.

ϕ \ ρ^*	0	1/6	1/4	1/3	5/12	1/2	7/12	2/3	3/4	10/12	11/12	1.00
$\pi/3$	1	1.000	1.001	1.001	1.002	1.004	1.006	1.009	1.013	1.019	1.029	1.050
$11\pi/36$	1	1.000	1.001	1.001	1.002	1.004	1.006	1.010	1.014	1.019	1.026	1.026
$5\pi/18$	1	1.000	1.001	1.002	1.003	1.005	1.007	1.009	1.011	1.009	1.003	0.996
$\pi/4$	1	1.000	1.001	1.002	1.003	1.004	1.003	1.000	0.994	0.986	0.979	0.971
$2\pi/9$	1	1.000	1.001	1.001	1.000	0.997	0.991	0.983	0.976	0.968	0.962	0.955
$7\pi/36$	1	1.000	1.000	0.998	0.993	0.986	0.979	0.971	0.965	0.959	0.954	0.949
$\pi/6$	1	0.999	0.997	0.992	0.986	0.979	0.972	0.967	0.962	0.958	0.955	0.953
$5\pi/36$	1	0.998	0.994	0.988	0.981	0.976	0.971	0.968	0.966	0.965	0.965	0.965
$\pi/9$	1	0.997	0.992	0.985	0.980	0.977	0.975	0.974	0.975	0.976	0.978	0.982
$\pi/12$	1	0.996	0.990	0.985	0.982	0.980	0.981	0.983	0.986	0.989	0.994	0.999
$\pi/18$	1	0.995	0.989	0.985	0.984	0.984	0.987	0.990	0.995	1.001	1.007	1.014
$\pi/36$	1	0.994	0.989	0.986	0.986	0.987	0.991	0.996	1.002	1.009	1.016	1.025
0	1	0.994	0.989	0.986	0.986	0.989	0.993	0.998	1.004	1.012	1.020	1.028

In order to verify the validity of the above speculations, we have performed some computations for a ’shamrock’-shaped punch by using an iterative procedure. The results are presented below.

ratio a_1/a_2	1.1	1.5	2.0
g (36)	0.4270	0.3971	0.3810
g numerical	0.4147	0.4013	0.3890
error of (36) (%)	3.0	1.0	2.0

As we can see, the numerical data do support all the above considerations. We note also that in the limiting case of $a_1/a_2\to\infty$, formula (36) gives the exact result for a circle, as it should.

Discussion. The examples presented show that the subroutine can be successfully used for solving various problems for non-classical domains. The applications are not limited to contact problems since the integral of the type (2) is involved in fluid mechanics, heat transfer, electrostatics, wave propagation *etc.* In the elastic contact problems, the subroutine helped to establish a very simple relationship between the applied force and the punch settlement (30–31) and to confirm its accuracy.

The method of this section can be successfully used for the computer evaluation of singular integrals of various types. For example, recently we have found the following integral representation for the reciprocal distance

$$\frac{1}{[\,\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\,]^{1/2}}=\frac{1}{\pi^2}\int_0^{2\pi}d\psi\int_0^{min(\rho_0,\rho)}\frac{(\rho^2-x^2)^{1/2}}{\rho^2+x^2-2x\rho\cos(\phi-\psi)}$$

$$\times\frac{(\rho_0^2-x^2)^{1/2}}{\rho_0{}^2+x^2-2x\rho_0\cos(\phi_0-\psi)}\,dx.$$

When $\rho<\rho_0$, the integrand in the last expression has a singularity at the boundary ($x=\rho$, $\psi=\phi$). Since the domain of integration in this case is a circle, the rhombi in Fig. 7.3.2 transform in two squares, and the following formula can be obtained for the integral estimation in the neighborhood of the singularity

$$I=\frac{1}{\pi^2}\left[\frac{2\varepsilon}{\rho}\right]^{1/2}[\,\pi(1+\sqrt{2})-2\sqrt{2}\ln(1+\sqrt{2})\,]\,\frac{(\rho_0{}^2-\rho^2)^{1/2}}{\rho^2+\rho_0{}^2-2\rho\rho_0\cos(\phi-\phi_0)}.$$

As it was noticed by Noble (1960), the existing methods generally provide the accuracy of about 5%. Of course, it would be very desirable to establish a Gauss type formula for the singular integral evaluation but so far no success was reported. Our subroutine can be used not only to verify the accuracy of an assumed solution but actually to find one. The first error in the suggested solution (22) is the assumption of a square root singularity at the edge which, in general, is incorrect. The second error is the assumption that c in (22) is a constant. It is of interest to establish the impact of each error on the accuracy of the solution. In order to do so, we assumed the solution in the form

$$\sigma(\rho,\phi)=\frac{c\,a^{1-u(\phi)}(\phi)}{[\,a^2(\phi)-\rho^2\,]^{[1-u(\phi)]/2}}. \tag{7.3.37}$$

The punch settlement at $\rho = 0$ will be defined by

$$w_0 = c\,\frac{\sqrt{\pi}}{2}\int_0^{2\pi} a(\phi)\frac{\Gamma[(1+u(\phi))/2]}{\Gamma[1+u(\phi)/2]}\,\mathrm{d}\phi. \qquad (7.3.38)$$

Certain modifications are to be made in formulae (6), (8), (19) and (18) which will take the respective forms

$$I_1(\rho,\phi) \approx 2\pi\varepsilon\,\frac{a^{1-u(\phi)}(\phi)\,f(\rho,\phi)}{[\,a^2(\phi)-\rho^2\,]^{[1-u(\phi)]/2}}, \qquad (7.3.39)$$

$$I_2(\rho,\phi) \approx \int_0^{2\pi} \frac{(2\varepsilon)^{[1+u(\phi_0)]/2}[a(\phi_0)]^{[3-u(\phi_0)]/2} f(a(\phi_0),\phi_0)\,\mathrm{d}\phi_0}{[1+u(\phi_0)][\,\rho^2 + a^2(\phi_0) - 2\rho a(\phi_0)\cos(\phi-\phi_0)]^{1/2}}, \qquad (7.3.40)$$

$$I_1(\phi) \approx (2\varepsilon)^{[1+u(\phi)]/2}[a(\phi)]^{[1-u(\phi)]/2} f(a(\phi),\phi)\cos\gamma\,[\,C(\gamma_+) + C(\gamma_-)], \qquad (7.3.41)$$

$$C(\gamma) = \int_0^{\pi/4-\gamma/2} \frac{\mathrm{d}\psi}{\sin^{[1-u(\phi)]/2}\psi\,\cos^{[1+u(\phi)]/2}(\psi+\gamma)} + \ln\frac{1+\cos\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)}.$$

The function $u(\phi)$ for the case of a square was taken $u(\phi) = 0.04 - 0.3(4\phi/\pi)^5$ for $0<\phi<\pi/4$. The pattern is repeated for $\phi>\pi/4$. The results of computations are presented in Table 7.3.4 which compares very favourably with Table 7.3.1 since now the normal displacements inside the square are practically uniform, except for a small portion close to the boundary where the maximum error is about 5%.

Table 7.3.4. Influence of the singularity on the displacements(square punch).

y^* \ x^*	0.000	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	1.000
1.000	0.999	0.997	0.995	0.995	0.996	1.001	1.013	1.030	1.050	1.053	0.978
0.950	1.006	1.004	1.002	1.000	0.999	1.000	1.002	1.004	1.000	0.995	1.053
0.900	1.006	1.004	1.002	1.000	0.999	0.998	0.997	0.995	0.991	1.000	1.050
0.800	1.006	1.004	1.002	1.000	0.997	0.995	0.992	0.989	0.995	1.004	1.030
0.700	1.005	1.003	1.001	0.998	0.996	0.993	0.991	0.992	0.997	1.002	1.013
0.600	1.003	1.002	1.000	0.997	0.995	0.993	0.993	0.995	0.998	1.000	1.001
0.500	1.002	1.001	0.999	0.997	0.996	0.995	0.996	0.997	0.999	0.999	0.996
0.400	1.001	1.000	0.999	0.998	0.997	0.997	0.998	1.000	1.000	1.000	0.995
0.300	1.001	1.000	0.999	0.999	0.999	1.000	1.001	1.002	1.002	1.002	0.995
0.200	1.000	1.000	1.000	1.000	1.001	1.002	1.003	1.004	1.004	1.004	0.997

0.100	1.000	1.000	1.000	1.001	1.002	1.003	1.004	1.005	1.006	1.005	0.999
0.000	1.000	1.000	1.001	1.001	1.002	1.003	1.005	1.006	1.006	1.006	0.999

Evaluation of (38) gives g = 0.3675, formula (33) gives the lower bound $g_e \geq 0.3666$, both results being very close to the value given by Noble. One can draw the conclusion: correct estimation of the singularities is the most important thing; the variation of c inside the square has a minor impact and is limited to, may be, several percent. The other idea which comes to mind is the possibility of the development of self-adaptive subroutine which would be able to find a solution with a prescribed degree of accuracy using the feedback principle. The first version of such a procedure has been developed. It takes (22) as initial stress distribution. The value of $c(\rho, \phi)$ is being adjusted at each point proportionally to the discrepancy between the computed potential and unity. The results of application of the procedure to a rectangle are given in Table 7.3.5.

Table 7.3.5. The normal displacements after 10 iterations (rectangle 1:10).

y^* \ x^*	0.00000	2.00000	3.00000	4.00000	5.00000	6.00000	7.00000	8.00000	9.00000	9.50000	10.00000
1.00000	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00001	1.00001	1.00003	1.00005	1.00000
0.95000	0.99996	1.00001	0.99992	0.99979	0.99947	0.99926	0.99823	0.99783	0.99538	0.99489	1.00000
0.90000	1.00001	0.99997	0.99983	0.99966	0.99923	0.99891	0.99776	0.99691	0.99504	0.99368	1.00000
0.80000	1.00002	0.99993	0.99976	0.99951	0.99906	0.99840	0.99756	0.99639	0.99615	0.99911	0.99999
0.70000	1.00002	0.99990	0.99973	0.99942	0.99894	0.99837	0.99749	0.99705	0.99892	1.00049	0.99999
0.60000	1.00001	0.99988	0.99971	0.99941	0.99897	0.99845	0.99810	0.99892	1.00046	1.00119	0.99999
0.50000	1.00001	0.99988	0.99974	0.99943	0.99912	0.99901	0.99951	1.00035	1.00103	1.00109	0.99999
0.40000	1.00001	0.99988	0.99986	0.99959	0.99963	1.00011	1.00066	1.00118	1.00119	1.00088	0.99999
0.30000	1.00000	0.99990	0.99990	1.00002	1.00047	1.00100	1.00145	1.00165	1.00115	1.00064	1.00000
0.20000	1.00000	0.99997	1.00019	1.00060	1.00111	1.00161	1.00193	1.00190	1.00105	1.00045	1.00000
0.10000	1.00000	1.00015	1.00051	1.00098	1.00149	1.00196	1.00219	1.00201	1.00097	1.00033	1.00000
0.00000	1.00000	1.00024	1.00062	1.00111	1.00162	1.00207	1.00227	1.00204	1.00094	1.00030	1.00000

Appendix

```
      FUNCTION  SING(F,A,R,FI,E,ER,IE)
      IMPLICIT  REAL*8  (A-H,O-Z)
      EXTERNAL  F,A,AN,AK,FA,FB,RR1,RR2,ANK,AKK,AK1,AK2
      COMMON/A/  R0,F0,EP,EP1
      R0=R
      F0=FI
      PI=3.1415926536
      EP1=.01
      IF(R-A(FI).GT..00001)  GO  TO  2
      AA=0.
      P2=2.*PI
      IF(R.LE.EP)  GO  TO  1
      AL=ASIN(EP/R)
      BB=FI-AL-EP1
      BB1=FI-AL
      CC1=FI+AL
```

```
      AA1=P2+FI−AL
      CC=FI+AL+EP1
      AA=P2+FI−AL−EP1
      IF(A(FI)−R.LE.EP)  GO  TO  3
      R1=DBLIN(FA,CC,AA,AN,AK,E,ER,IE)
      R21=DBLIN(FA,BB,BB1,AN,AK1,E,ER,IE)
      R22=DBLIN(FA,CC1,CC,AN,AK1,E,ER,IE)
      R23=DBLIN(FA,BB,BB1,AK2,AK,E,ER,IE)
      R24=DBLIN(FA,CC1,CC,AK2,AK,E,ER,IE)
      R25=DBLIN(FA,BB,BB1,AK1,AK2,E,ER,IE)
      R26=DBLIN(FA,CC1,CC,AK1,AK2,E,ER,IE)
      R2=R21+R22+R23+R24+R25+R26
      R3=DBLIN(FA,BB1,CC1,AN,RR1,E,ER,IE)
      R4=DBLIN(FA,BB1,CC1,RR2,AK,E,ER,IE)
      R5=P2*EP*F(FI,R)/SQRT(A(FI)**2−R*R)*A(FI)
      R6=DCADRE(FB,0.,P2,E,E,ER,IE)
    5 FORMAT(1X,6F10.5)
      SING=R1+R2+R3+R4+R5+R6
      GO  TO  4
    1 R1=DBLIN(FA,AA,P2,ANK,AK,E,ER,EI)
      R5=P2*EP*F(FI,0.)
      R6=DCADRE(FB,0.,P2,E,E,ER,IE)
      SING=R1+R5+R6
      GO  TO  4
    2 TYPE*,'OUTSIDE'
      R=A(FI)
      R1=DBLIN(FA,AA,P2,AN,AK,E,ER,IE)
      R6=DCADRE(FB,0.,P2,E,E,ER,IE)
      SING=R1+R6
    6 FORMAT(1X,7HOUTSIDE)
      GO  TO  4
    3 DA=(A(FI+EP)−A(FI))/EP
      GP=−ASIN(DA/SQRT(A(FI)**2+DA**2))
      DA=(A(FI)−A(FI−EP))/EP
      GM=ASIN(DA/SQRT(A(FI)**2+DA**2))
      AL1=ASIN(EP*COS(GP)/R)+FI
      AL2=FI−ASIN(EP*COS(GM)/R)
      BB=AL2−EP1
      BB1=AL2
      CC1=AL1
      AA1=P2+AL2
      CC=AL1+EP1
      AA=P2+AL2−EP1
      R1=DBLIN(FA,CC,AA,AN,AK,E,ER,IE)
      R5=(FG(GP)+FG(GM))
      R21=DBLIN(FA,BB,BB1,AN,AKK,E,ER,IE)
      R22=DBLIN(FA,CC1,CC,AN,AKK,E,ER,IE)
      BB2=BB+.99*EP1
      R23=DBLIN(FA,BB,BB2,AKK,AK,E,ER,IE)
      R231=DBLIN(FA,BB2,BB1,AKK,AK,E,ER,IE)
      R24=DBLIN(FA,CC1,CC,AKK,AK,E,ER,IE)
      R2=R21+R22+R23+R24+R231
      R3=DBLIN(FA,BB1,CC1,AN,AKK,E,ER,IE)
      R4=DBLIN(FA,BB1,CC1,AKK,AK,E,ER,IE)
```

```
      R6=DCADRE(FB,CC,AA,E,E,ER,IE)
      1+DCADRE(FB,AL1,CC,E,E,ER,IE)
      2+DCADRE(FB,BB,AL2,E,E,ER,IE)
      SING=R1+R2+R3+R4+R6+R5
    4 RETURN
      END
C     *******************************
      FUNCTION  F2(XF)
      IMPLICIT  REAL*8  (A–H,O–Z)
      COMMON/A/  R,FI,EP,EP1
      DIMENSION  XX(12),YY(12),C(2,12,2,12)
      COMMON/E/  C,XX,YY,CS
      XL=XX(12)
      YL=XF
      IF(ABS(YY(12)–YL).LT..0000001)  YL=YY(12)
      IF(ABS(YY(1)–YL).LT..0000001)  YL=YY(1)
      CALL  IBCEVL(XX,12,YY,12,CS,12,XL,YL,FL,IE)
    5 FORMAT(1X,5F15.4)
      Z=1.+U(XF)
      F2=FL*A(XF)**2*SIN(EP)**Z/Z
      RETURN
      END
C     ********************************
      FUNCTION  FU(FI,PS)
      IMPLICIT  REAL*8  (A–H,O–Z)
      DIMENSION  XX(12),YY(12),C(2,12,2,12)
      COMMON/E/  C,XX,YY,CS
      XL=SIN(PS)
      YL=FI
      IF(ABS(XX(1)–XL).LT..0000001)  XL=XX(1)
      IF(ABS(YY(12)–YL).LT..0000001)  YL=YY(12)
      IF(ABS(XX(12)–XL).LT..0000001)  XL=XX(12)
      IF(ABS(YY(1)–YL).LT..0000001)  YL=YY(1)
      CALL  IBCEVL(XX,12,YY,12,CS,12,XL,YL,FL,IE)
    5 FORMAT(1X,5F15.4)
      FU=FL*A(FI)**2*SIN(PS)*COS(PS)**U(FI)
      RETURN
      END
C     **************************************
      FUNCTION  FU1(FI,PS)
      IMPLICIT  REAL*8  (A–H,O–Z)
      DIMENSION  XX(12),YY(12),C(2,12,2,12)
      COMMON/E/  C,XX,YY,CS
      XL=SIN(PS)
      YL=FI
      IF(ABS(XX(1)–XL).LT..0000001)  XL=XX(1)
      IF(ABS(YY(12)–YL).LT..0000001)  YL=YY(12)
      IF(ABS(XX(12)–XL).LT..0000001)  XL=XX(12)
      IF(ABS(YY(1)–YL).LT..0000001)  YL=YY(1)
      CALL  IBCEVL(XX,12,YY,12,C,12,XL,YL,FL,IE)
      CALL  IBCEVL(XX,12,YY,12,CS,12,XL,YL,FLS,IE)
    5 FORMAT(1X,5F15.4)
      IF(COS(PS).LE.0.)  TYPE*,'FU1',PS,COS(PS),FI
      FU1=FL*FLS*A(FI)**2*SIN(PS)*COS(PS)**U(FI)
```

```
      RETURN
      END
C     ****************************************
      FUNCTION FU2(XF)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      DIMENSION XX(12),YY(12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      XL=XX(12)
      YL=XF
      IF(ABS(YY(12)–YL).LT..0000001) YL=YY(12)
      IF(ABS(YY(1)–YL).LT..0000001) YL=YY(1)
      CALL IBCEVL(XX,12,YY,12,C,12,XL,YL,FL,IE)
      CALL IBCEVL(XX,12,YY,12,CS,12,XL,YL,FLS,IE)
    5 FORMAT(1X,5F15.4)
      Z=1.+U(XF)
      FU2=FL*FLS*A(XF)**2*SIN(EP)**Z/Z
      RETURN
      END
C     ******************************************
      FUNCTION F1(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      F1=A(X)*GAMMA((1.+U(X))/2.)/GAMMA(1.+U(X)/2.)
      RETURN
      END
C     *****************************************
      FUNCTION APE(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      APE=3.1415926536/2.–EP
      RETURN
      END
C     *******************************************
      FUNCTION AP2(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      AP2=3.1415926536/2.
      RETURN
      END
C     *******************************************
      FUNCTION F(F0,X)
      IMPLICIT REAL*8 (A–H,O–Z)
      DIMENSION XX(12),YY(12),CS(2,12,2,12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      PI=3.1415926536
      FI=F0
    1 IF(FI.LT.0.) FI=FI+2.*PI
    2 IF(FI.GT.PI/3.) FI=FI–2.*PI/3.
      IF(FI.GT.PI/3.) GO TO 2
      XL=X/A(FI)
      YL=ABS(FI)
      IF(ABS(XX(1)–XL).LT..00001) XL=XX(1)
      IF(ABS(YY(12)–YL).LT..00001) YL=YY(12)
      IF(ABS(XX(12)–XL).LT..00001) XL=XX(12)
      IF(ABS(YY(1)–YL).LT..00001) YL=YY(1)
```

```
    5 FORMAT(1X,5F15.4)
      CALL  IBCEVL(XX,12,YY,12,CS,12,XL,YL,FL,IE)
      IF(IE.NE.0)  TYPE*,'F  ER',IE,XL,YL,FL
      F=FL
      RETURN
      END
C     ****************************************
      FUNCTION  U(X)
      IMPLICIT  REAL*8  (A-H,O-Z)
      DIMENSION  UD(23),PSI(23),CRR(22,3)
      COMMON/D/  UD,PSI,CRR
      PI=3.1415926536
      Y=ABS(X)
    1 IF(Y.GT.PI/3.)  Y=ABS(Y-2.*PI/3.)
      IF(Y.GT.PI/3.)  GO  TO  1
      CALL  ICSEVU(PSI,UD,23,CRR,22,Y,RES,1,IER)
      IF(IER.NE.0)  TYPE*,'U  ER',X,Y,RES
    5 FORMAT(1X,3F15.4)
      U=RES
      RETURN
      END
C     ****************************************
      FUNCTION  AKK(X)
      IMPLICIT  REAL*8  (A-H,O-Z)
      COMMON/A/  R,FI,EP,EP1
      AKK=A(X)-EP-EP1
      RETURN
      END
C     ****************************************
      FUNCTION  AK(X)
      IMPLICIT  REAL*8  (A-H,O-Z)
      COMMON/A/  R,FI,EP,EP1
      AK=A(X)-EP
      RETURN
      END
C     ****************************************
      FUNCTION  RR1(X)
      IMPLICIT  REAL*8  (A-H,O-Z)
      COMMON/A/  R,FI,EP,EP1
      SQ=(EP**2-(R*SIN(FI-X))**2)
      IF(SQ.LT.0.)  SQ=0.
      RR1=R*COS(FI-X)-SQRT(SQ)
      IF(RR1.GT.AK(X))  RR1=AK(X)
      RETURN
      END
C     ****************************************
      FUNCTION  RR2(X)
      IMPLICIT  REAL*8  (A-H,O-Z)
      COMMON/A/  R,FI,EP,EP1
      SQ=(EP**2-(R*SIN(FI-X))**2)
      IF(SQ.LT.0.)  SQ=0.
      RR2=R*COS(FI-X)+SQRT(SQ)
      IF(RR2.GT.AK(X))  RR2=AK(X)
      RETURN
```

```
      END
C     ****************************************
      FUNCTION AN(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      AN=0.
      RETURN
      END
C     ***************************************
      FUNCTION A(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/B/ A1,A2,B
      A=A1+A2*COS(3.*X)
      RETURN
      END
C     ************************************************
      FUNCTION MMDELK(Z)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/MM/ ARG
      REAL*8 MMDELK
      MMDELK=1./SQRT(1.–(ARG*SIN(Z))**2)
      RETURN
      END
C     ************************************************
      FUNCTION FG(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      COMMON/MM/ ARG
      EXTERNAL MMDELK
      REAL*8 MMDELK
      E=.00001
      PI=3.1415926536
      Y=PI/4.–X/2.
      ARG=SIN(Y)
      C=COS(Y)
      P2=PI/2.
      FG=DCADRE(MMDELK,0.,P2,E,E,ER,IE)+LOG((1.+C)/C)
      FG=FG*COS(X)*F(FI,A(FI))*SQRT(2.*A(FI)*EP)
      RETURN
      END
C     ***************************************************
      FUNCTION AK1(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      Y=R–EP1
      IF(Y.GT.AK(X))Y=AK(X)
      AK1=Y
      RETURN
      END
C     ************************************************
      FUNCTION AK2(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      Y=R+EP1
      IF(Y.GT.AK(X))Y=AK(X)
```

```
      AK2=Y
      RETURN
      END
C     ************************************************
      FUNCTION FA(PS,X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      RR=SQRT(R*R+X*X–2.*R*X*COS(FI–PS))
      SQ=SQRT(A(PS)**2–X*X)
      FA=F(PS,X)*(A(PS)/SQ)**(1.–U(PS))/RR*X
      RETURN
      END
C     ************************************************
      FUNCTION ANK(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      ANK=EP
      RETURN
      END
C     ***********************************************
      FUNCTION FB(PS)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      AA=A(PS)
      RR=SQRT(R*R+AA*AA–2.*R*AA*COS(FI–PS))
      Z=U(PS)
      FB=AA**(1.–Z)*(2.*EP*AA)**((1.+Z)/2.)/(1.+Z)*F(PS,AA)/RR
      RETURN
      END
```

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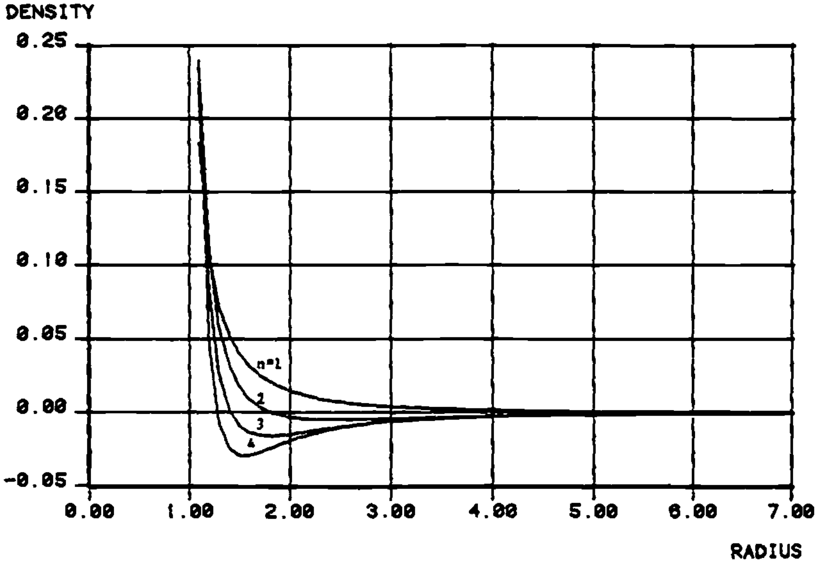
Subject Index

The page numbers in this section are approximate. They were correct in original publication, not so here.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0.10
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2
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3.00
4.00
5.00
6.00
7.00
RADIUS

… (formula 40) are presented in Fig. 1.4.2.

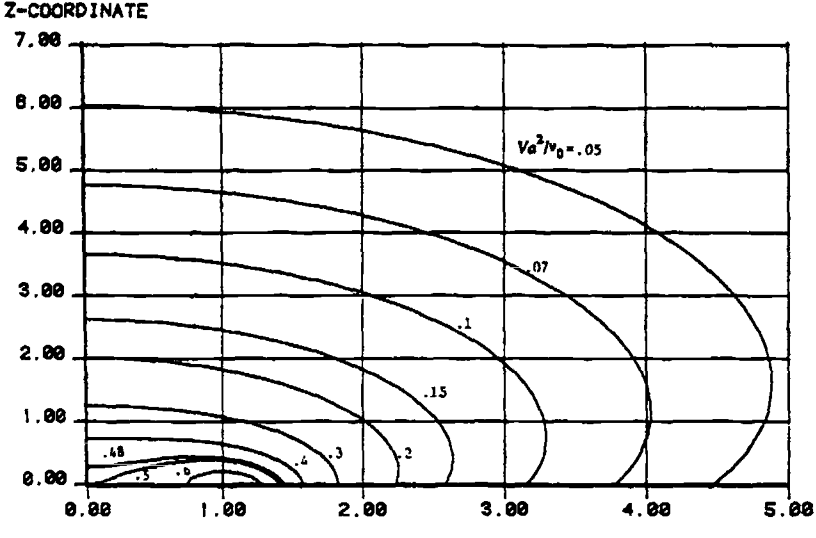


Fig. 1.4.2. Equipotential lines for $n=2$

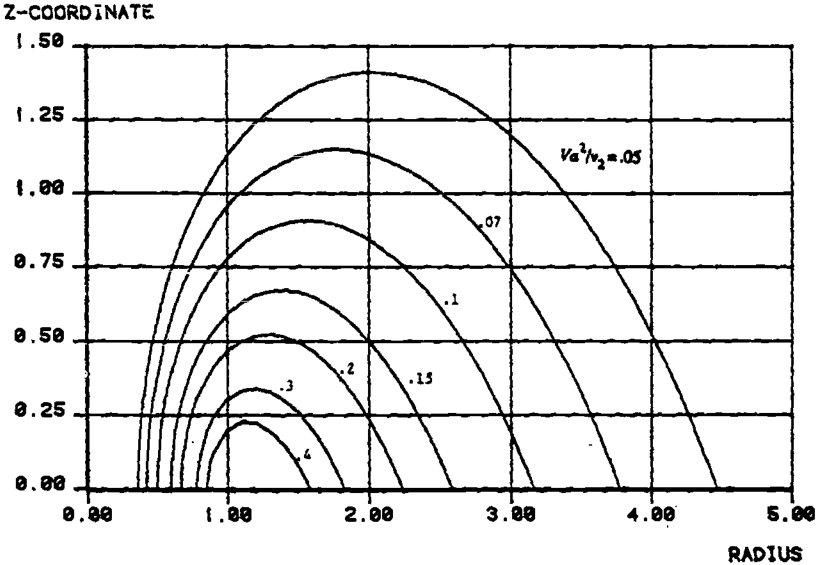


Fig. 1.4.3. Equipotential lines for $n=2$

Example 3. Consider a case related to Problem 2, with the boundary conditions

$$V=0, \text{ for } \rho \leq a,\ 0 \leq \phi < 2\pi;$$

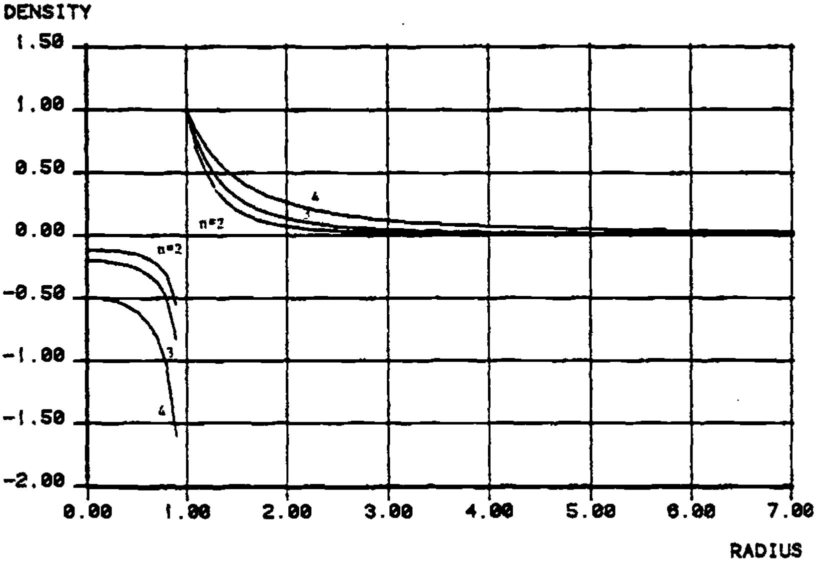


Fig. 1.4.4. Charge distribution for $n=2,3,4$

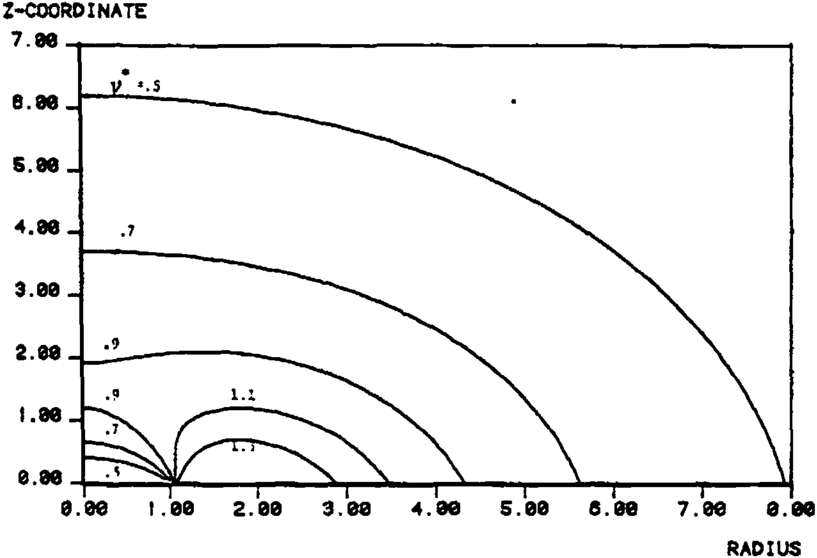


Fig. 1.4.5. Equipotential lines for $n=3$

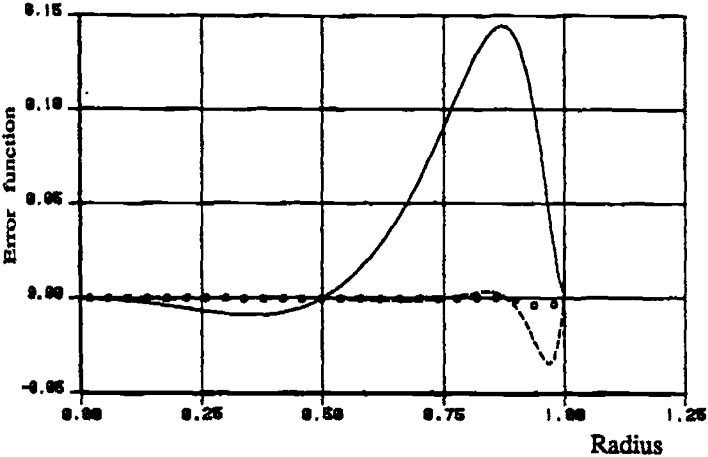


Fig. 3.1.1. Error function for equidistant points of collocations.

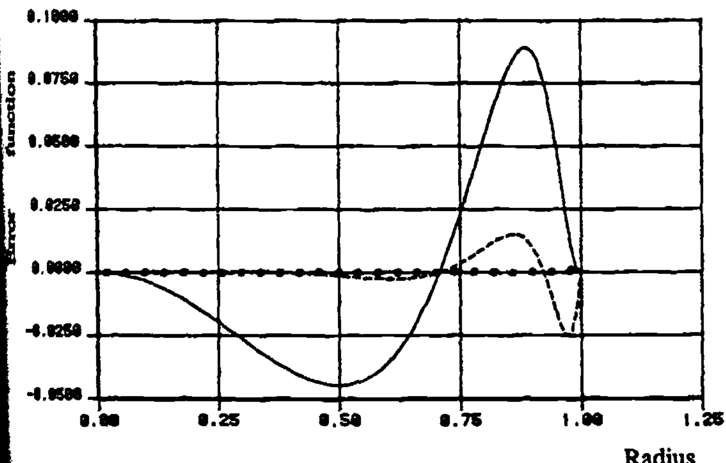
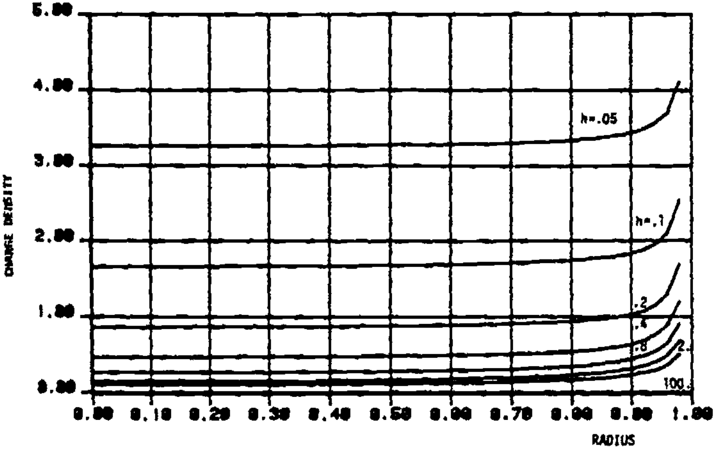


Fig. 3.1.2. Error function for non-equidistant points of collocations.



3.1.3. Charge density distribution (two disks at unit opposite potentials).

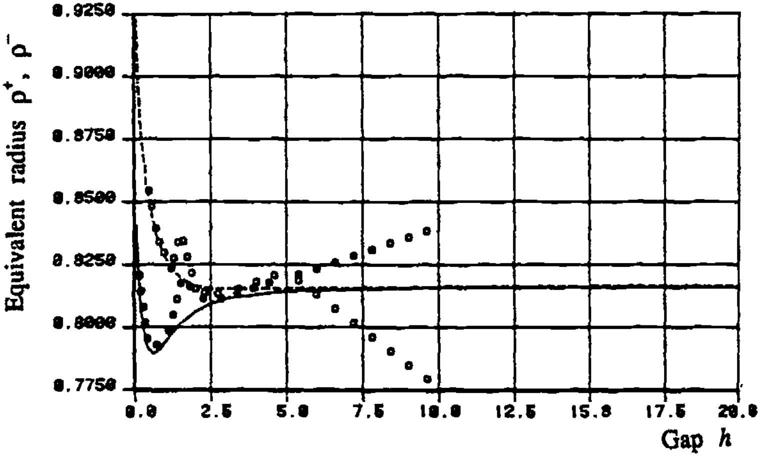


Fig. 3.1.4. Test of the accuracy of numerical results

and Fig. 3.1.6 respectively. The solid line in both figures gives

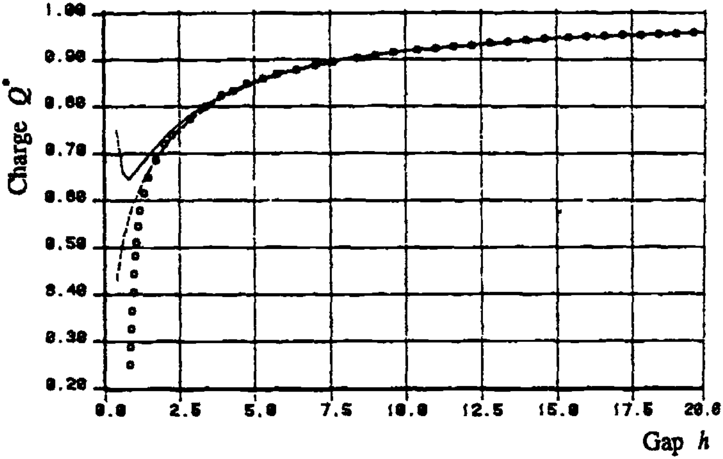


Fig. 3.1.5. Total charge at the first disk (three-disk system $v_1=v_2=v_3=v$)

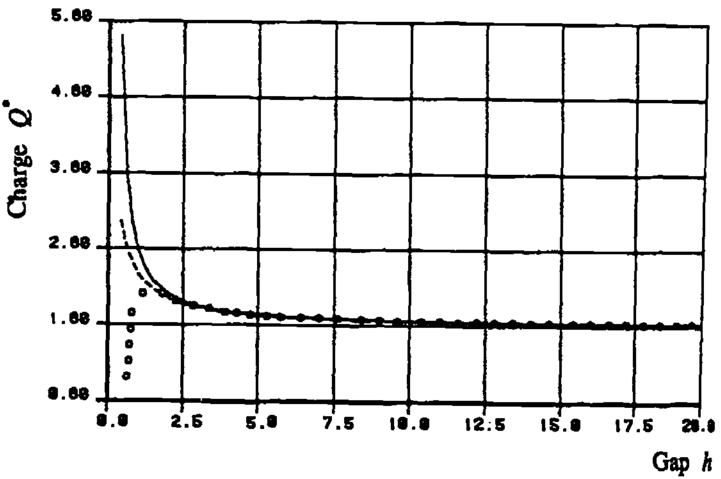


Fig. 3.1.6. Total charge at the first disk (three-disk system $v_1=-v_2=v$, $v_3=0$

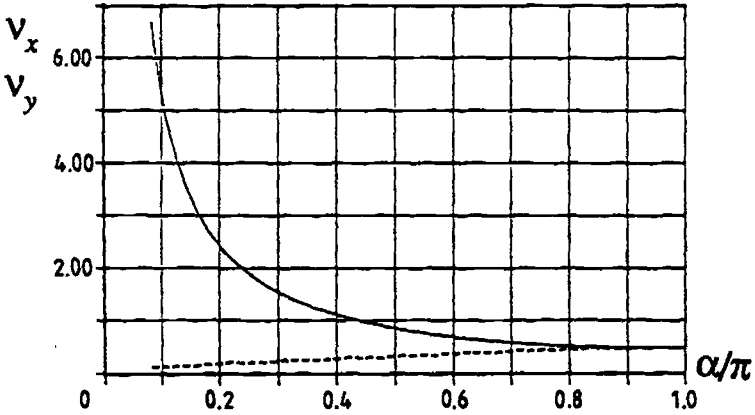


Fig. 3.4.1. Coefficients of magnetic polarizability for circular segment

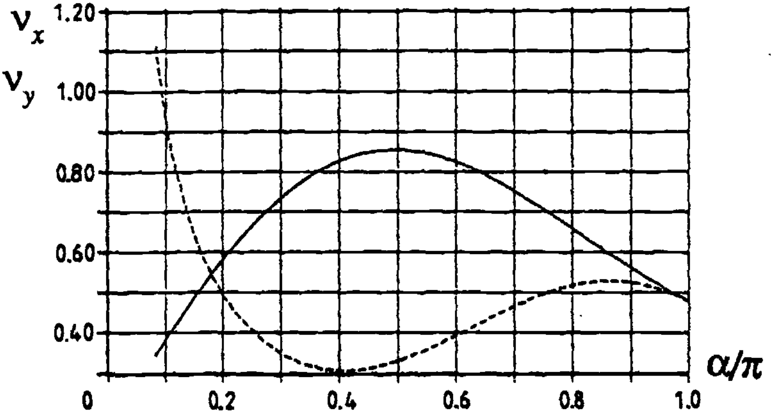


Fig. 3.4.2. Coefficients of magnetic polarizability for circular sector

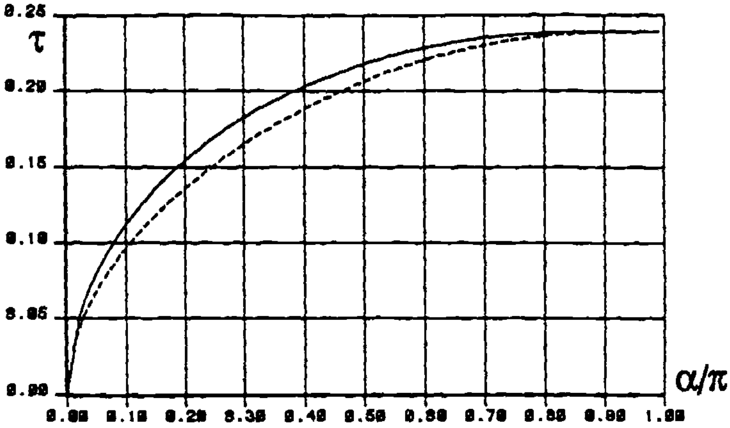


Fig. 3.5.1. Coefficient of electrical polarizability for circular segment

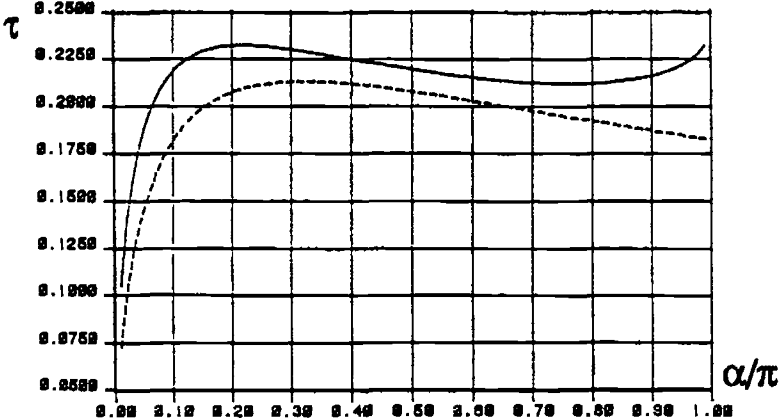


Fig. 3.5.2. Coefficient of electrical polarizability for circular sector

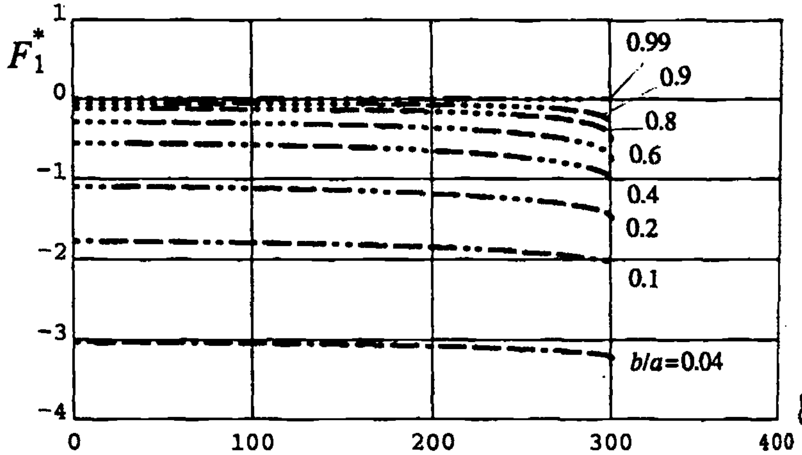


Fig. 3.7.1. Uniform charge distribution (solution for F_1)

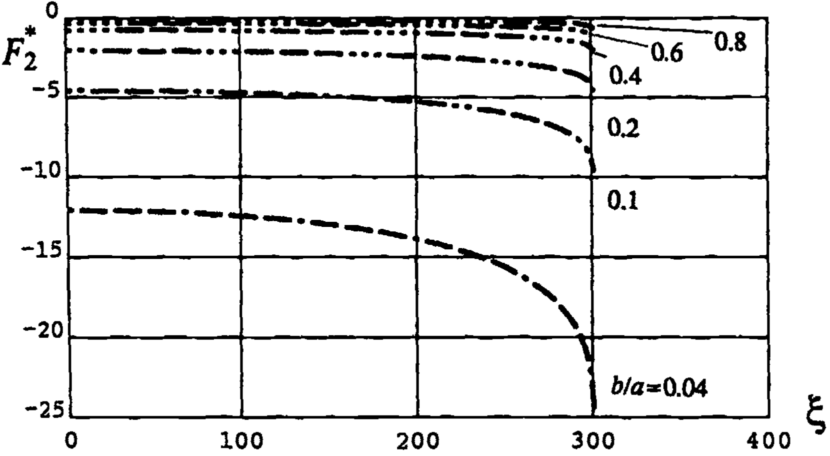


Fig. 3.7.2. Uniform charge distribution (solution for F_2)

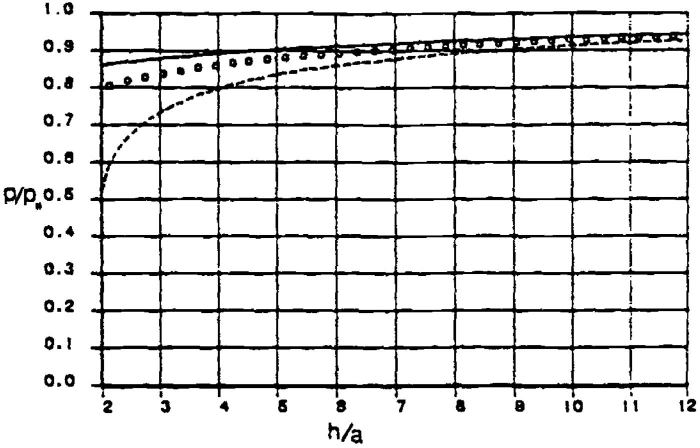


Fig. 4.1.1. The ratio P/P_0 for two equal elliptical holes

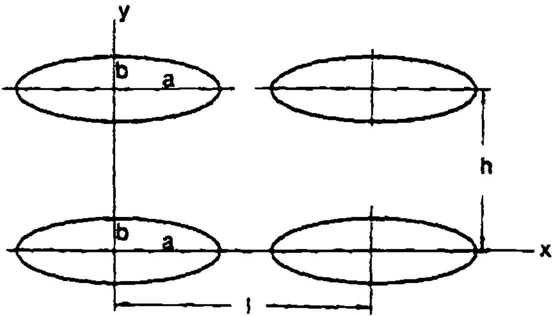


Fig. 4.1.2. Four equal elliptical holes

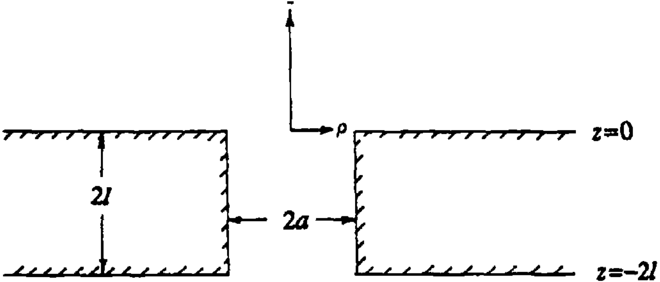


Fig. 4.3.1. Geometry of the problem

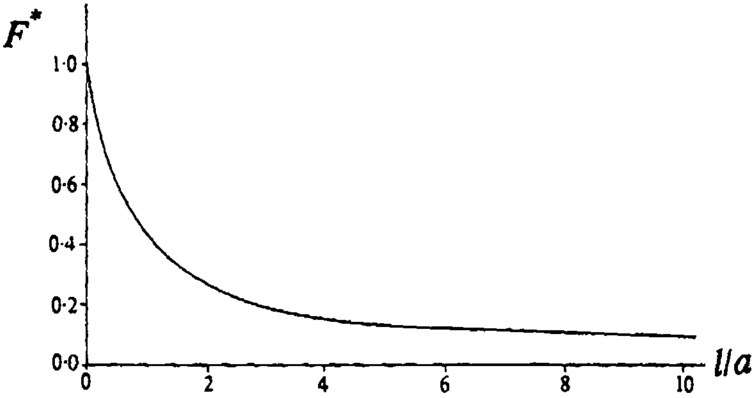


Fig. 4.3.2. Dimensionless flux through a pore of length l

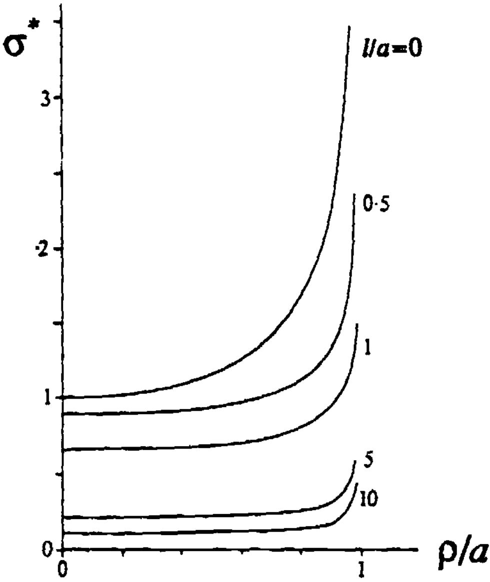


Fig. 4.3.3. Local flux at the pore entrance

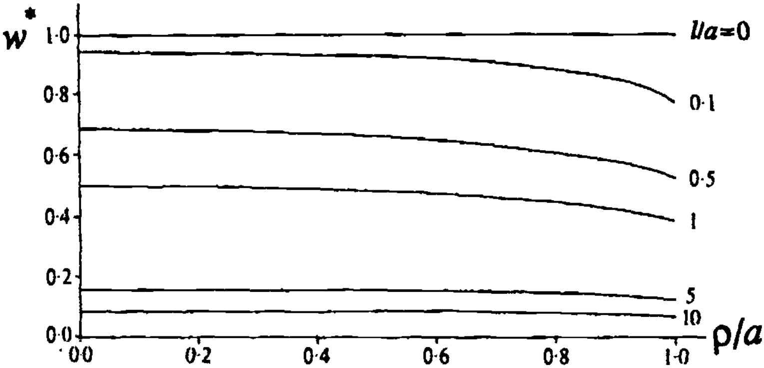
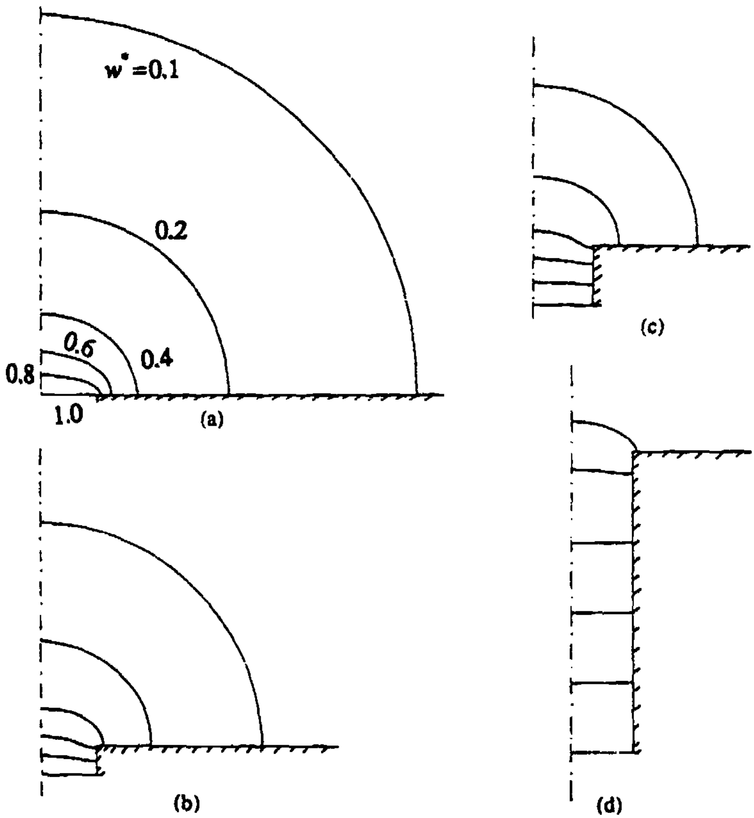


Fig. 4.3.4. Local concentration profile at the pore entrance



4.3.5. Isoconcentration profiles: (a)λ=0, (b)λ=0.5, (c)λ=1., (d)λ=5.

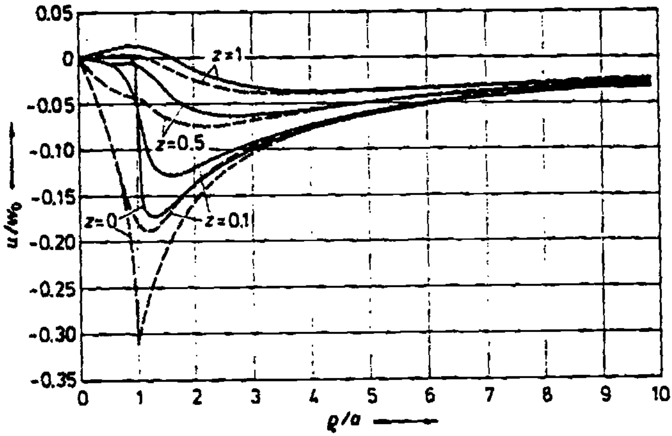


Fig. 5.1.1. The field of radial displacements

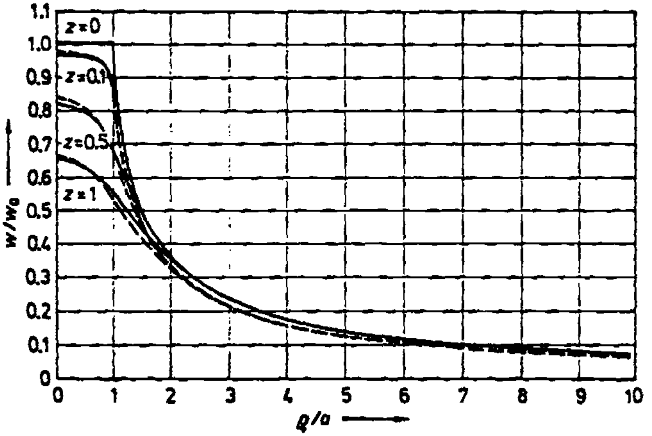


Fig. 5.1.2. The field of normal displacements

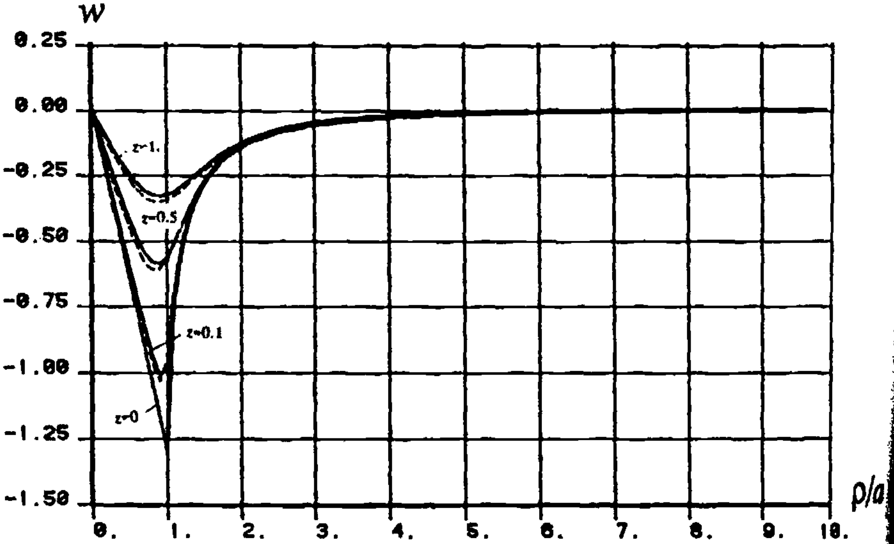


Fig. 5.2.1. The influence of bonding on normal displacements

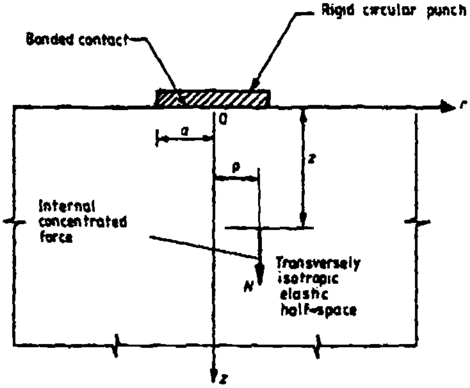


Fig. 5.3.1. Geometry of the problem

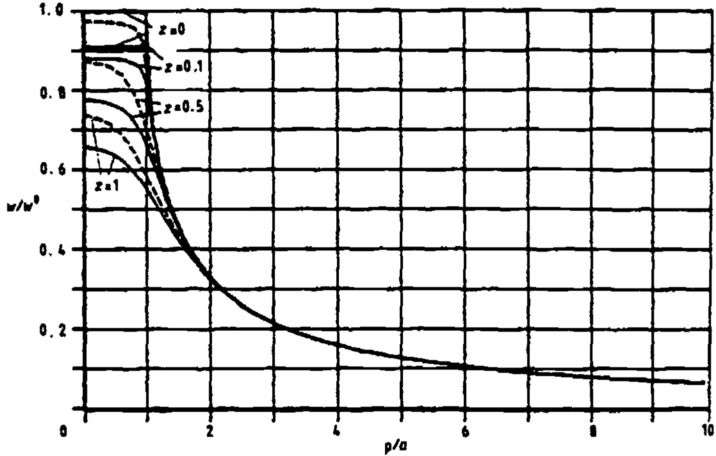


Fig. 5.3.2. Influence of bonding on the punch settlement

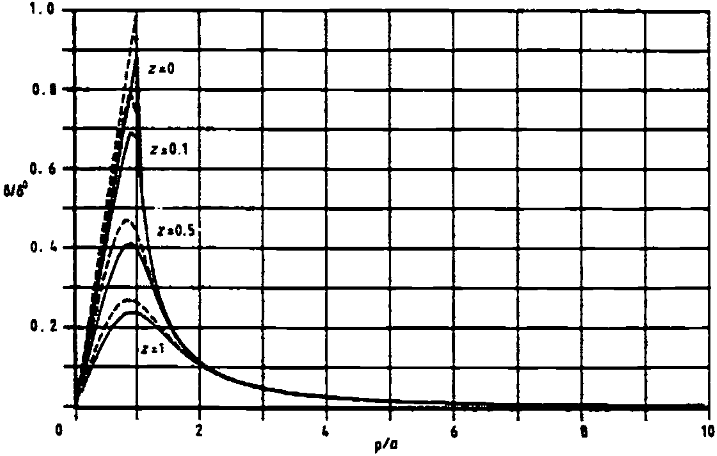


Fig. 5.3.3. Influence of bonding on angular inclination

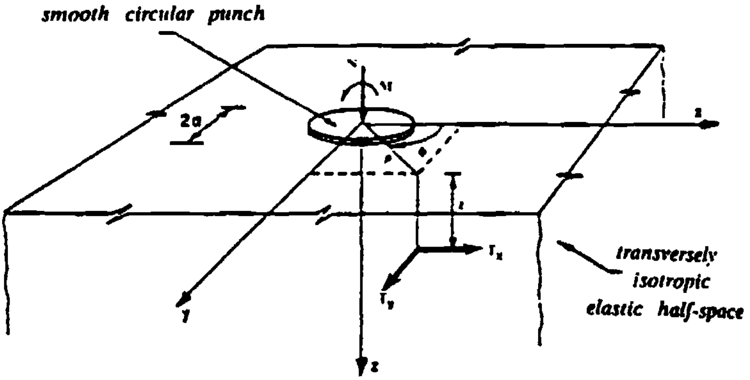


Fig. 5.4.1. Geometry of the problem

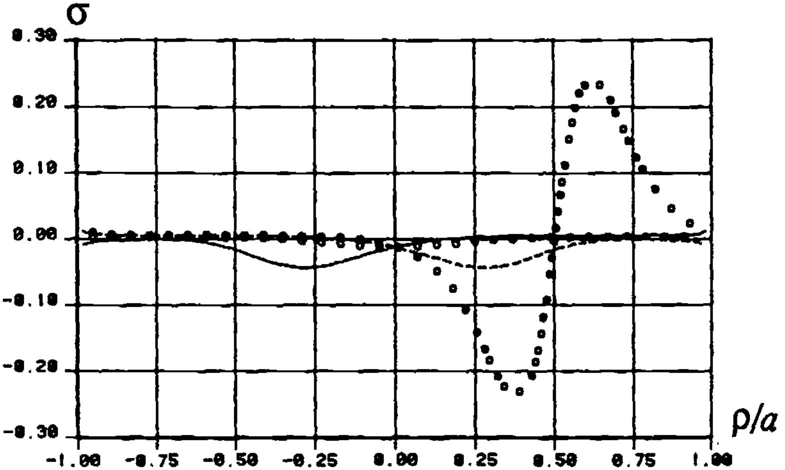


Fig. 5.4.2. Stress distribution due to a unit radial force

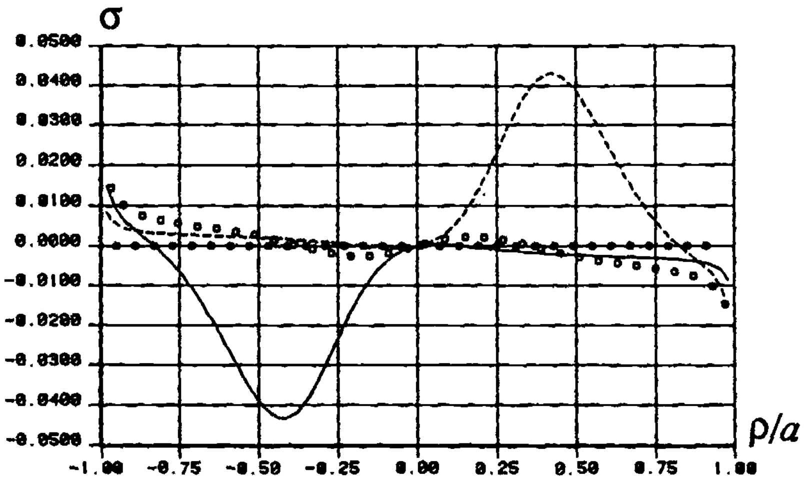


Fig. 5.4.3. Stress distribution due to a unit transversal force

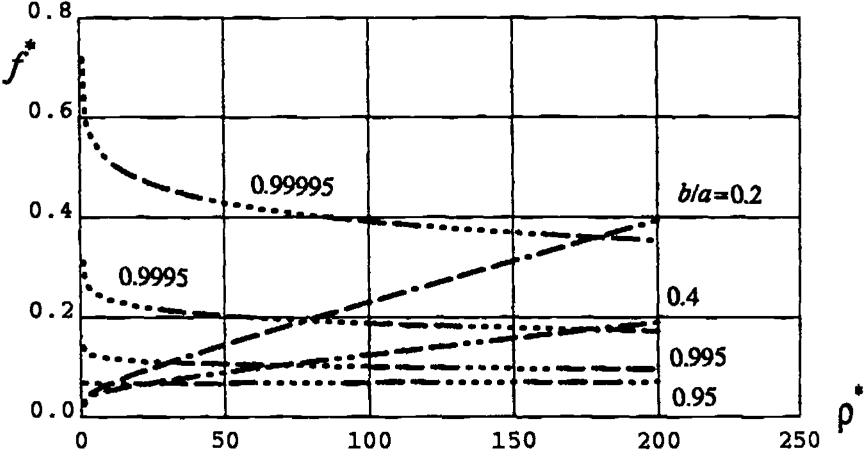


Fig. 5.5.1. Solution for a centrally loaded annular punch

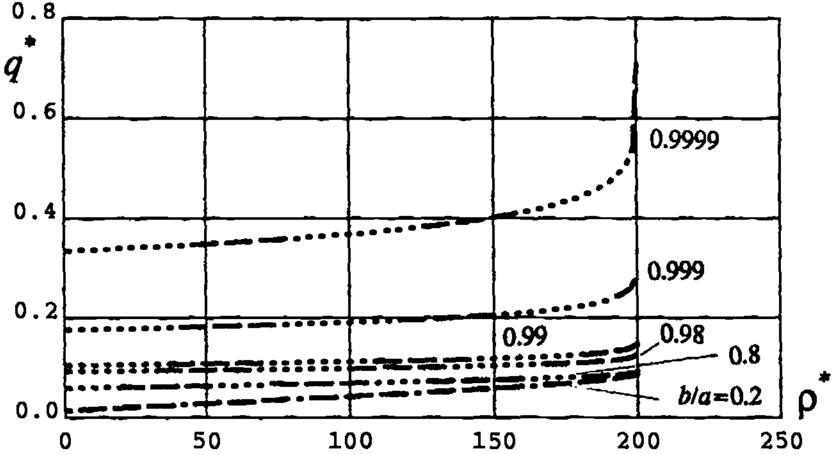


Fig. 5.5.2. Solution for an inclined annular punch

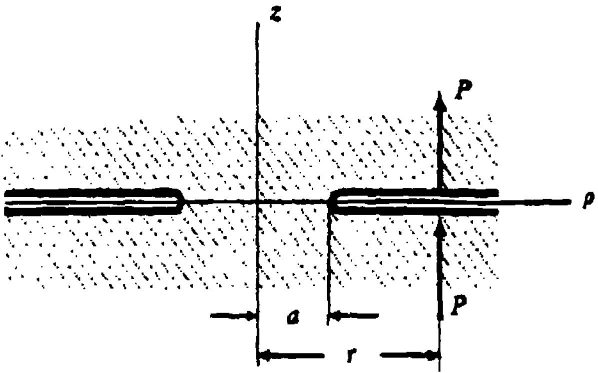


Fig. 6.1.1. The crack geometry

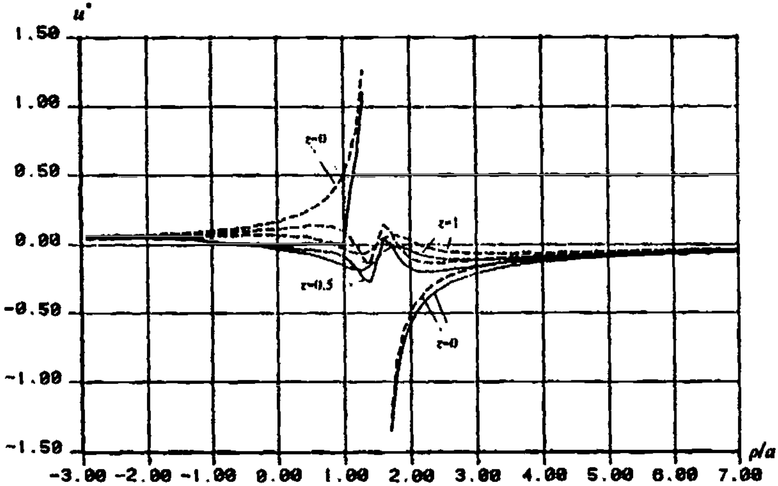


Fig. 6.1.2. The field of tangential displacements

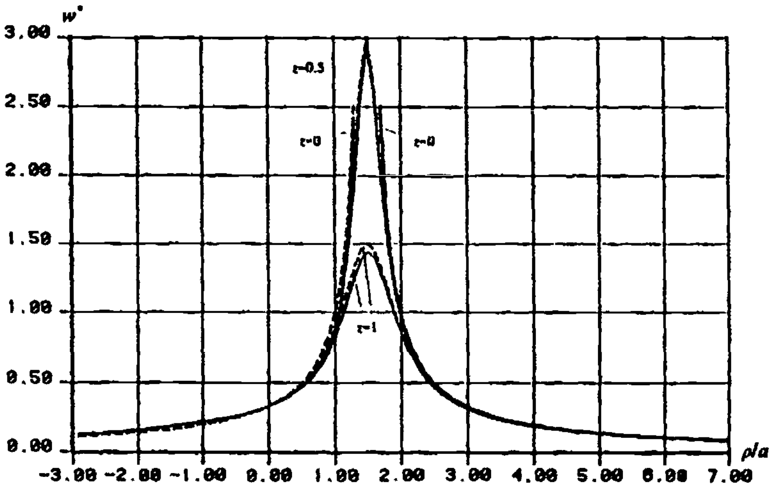


Fig. 6.1.3. The field of normal displacements

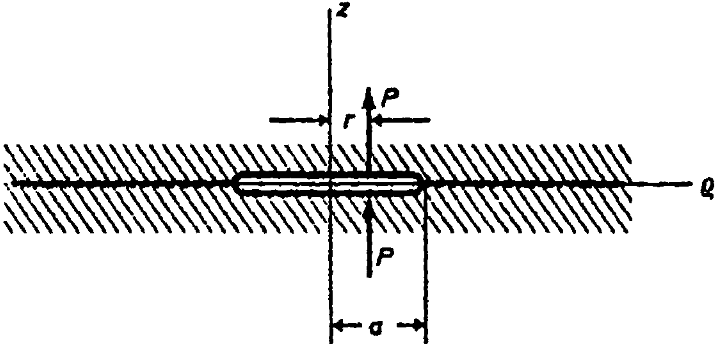


Fig. 6.2.1. Loading of a penny-shaped crack

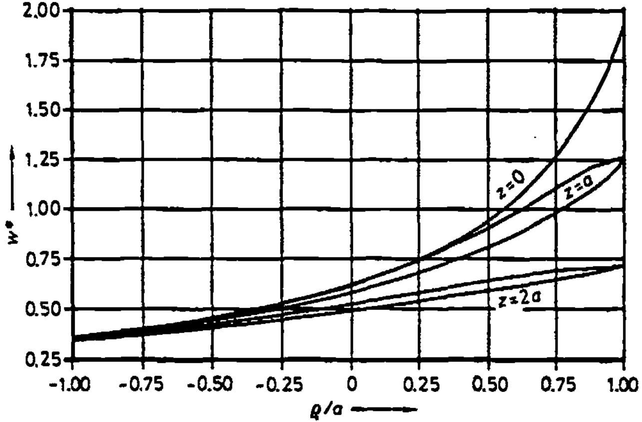


Fig. 6.2.2. Crack shape due to an external force

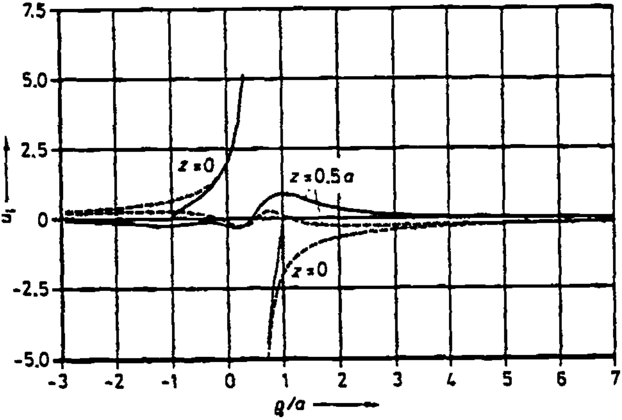


Fig. 6.2.3. Comparison of tangential displacements

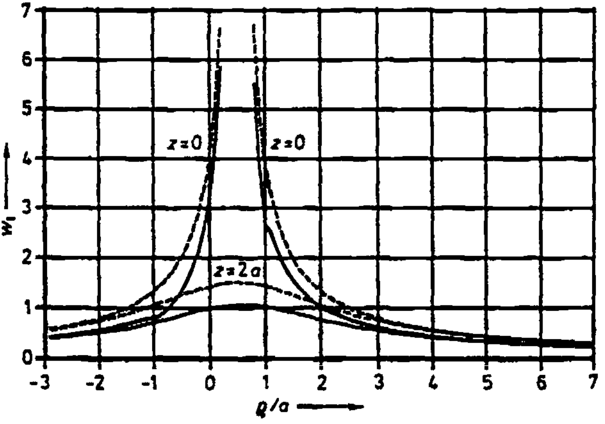


Fig. 6.2.4. Comparison of normal displacements

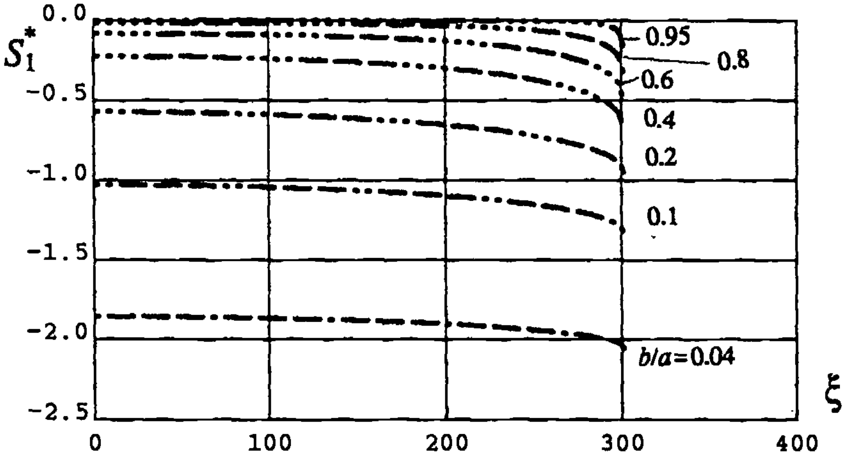


Fig. 6.3.1. Annular crack under uniform normal loading (solution for S_1)

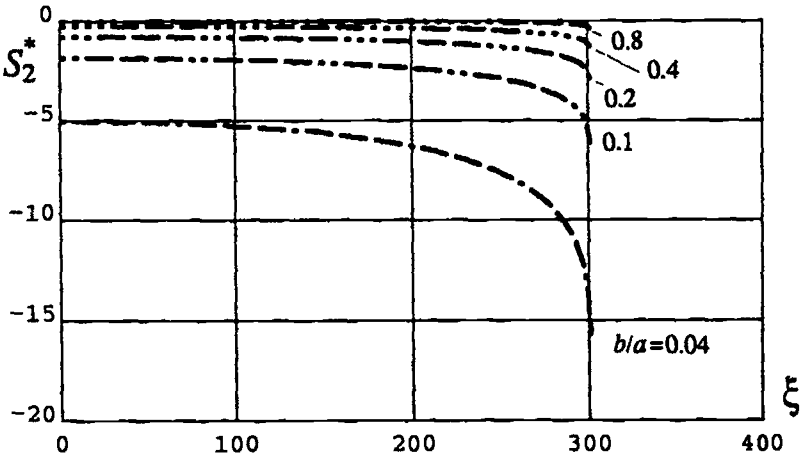


Fig. 6.3.2. Annular crack under uniform normal loading (solution for S_2)

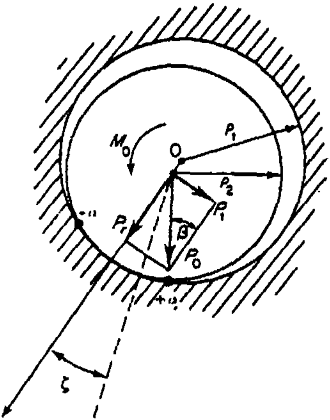


Fig. 7.2.1. Plane contact problem of a disk inserted in a plate

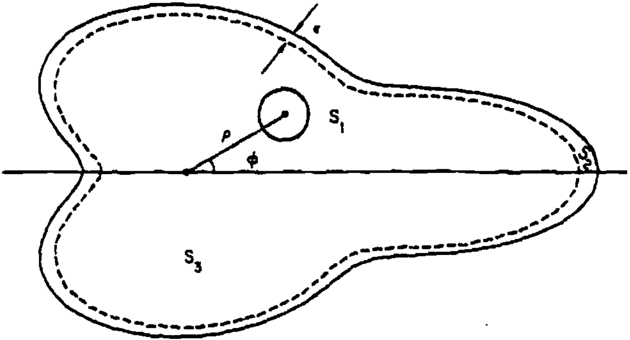


Fig. 7.3.1. Subdomains of integration

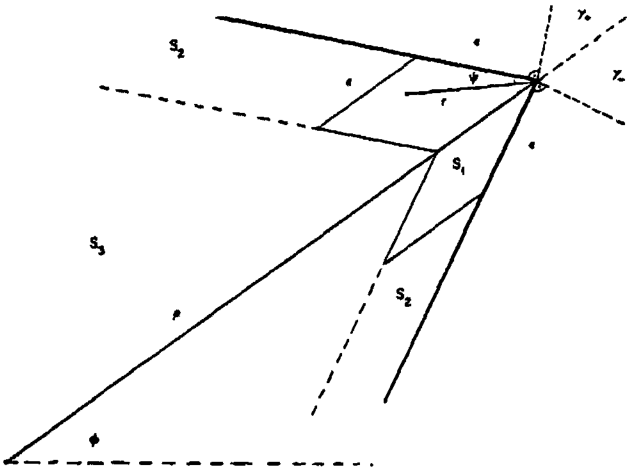


Fig. 7.3.2. Geometry related to the derivation of (19) and (20)